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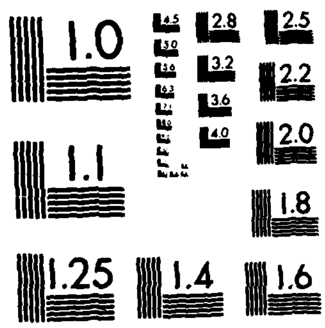

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STOCHASTIC REARRANGEMENT INEQUALITIES

by

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Stochastic Rearrangement Inequalities

by

Catherine D'Abadie and Frank Proschan

ABSTRACT

We develop a unified theory for obtaining stochastic rearrangement inequalities. We present sample applications in ranking problems, hypothesis testing, contamination models, optimal assembly of systems, and stochastic versions of well known rearrangement inequalities.

1. Introduction and Summary. We obtain stochastic versions of rearrangement inequalities. Rearrangement inequalities compare the value of a function of vector arguments with the value of the same function after the components of the vectors have been rearranged.

The classical example of a rearrangement inequality involving a function of 2 vector arguments is the well-known inequality of Hardy, Littlewood, Pólya (HLP) (1952) for sums of products. For vectors $\underset{\sim}{a} = (a_1,\ldots,a_n)$ and $\underset{\sim}{b} = (b_1,\ldots,b_n)$ of positive numbers, HLP show that if $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ (after relabeling, say), for every permutation $(\pi(1),\ldots,\pi(n))$ of $(1,\ldots,n)$, then

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\pi(i)} \geq \sum_{i=1}^{b} a_i b_{n-i+1} \tag{1.1}$$

hold.

What is a stochastic version of (1.1)? Under what conditions will

$$\sum_{i=1}^{n} X_i Y_i \overset{st.}{\geq} \sum_{i=1}^{n} X_i Y_{\pi(i)} \overset{st.}{\geq} \sum_{i=1}^{n} X_i Y_{n-i+1} \tag{1.2}$$

hold, given random vectors $\underset{\sim}{X}$, $\underset{\sim}{Y}$? To answer this question, we need the following definition:

1.1 Definition. For vector $\underset{\sim}{x}$ we write $\underset{\sim}{x} \overset{t_{ij}}{\geq} \underset{\sim}{x}'$ if $i < j$, $x_i \leq x_j$ and $\underset{\sim}{x}'$ is obtained from $\underset{\sim}{x}$ by interchanging x_i and x_j and leaving the other components fixed.

1.2 Proposition. Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be nonnegative random vectors having joint density $f(\underset{\sim}{x}\underset{\sim}{y})$. Then inequality (1.2) holds for $\underline{X}$ and $\underline{Y}$ if for all pairs (i,j), $1 \leq i < j \leq n$, f satisfies:

$$f(\underset{\sim}{x},\underset{\sim}{y}) + f(\underset{\sim}{x}',\underset{\sim}{y}') - f(\underset{\sim}{x}',\underset{\sim}{y}) - f(\underset{\sim}{x},\underset{\sim}{y}') \geq 0, \text{ where } \underline{x} \overset{t_{ij}}{\geq} x' \text{ and } \underline{y} \overset{t_{ij}}{\geq} y'. \tag{1.3}$$

Many well-known multivariate densities satisfy this condition, as we shall see.

Since the work of HLP, many papers on rearrangement inequalities have appeared. More recent are London (1970), Minc (1971), and Day (1972). The Marshall and

Olkin (MO) (1979) book presents a unified approach to the study of deterministic rearrangement inequalities.

In this paper we develop a unified theory for obtaining stochastic versions of rearrangement inequalities. (1.2) becomes a special case, as does the work of many earlier authors. Moreover, we obtain stochastic refinements of these inequalities analogous to those obtained by MO in the deterministic case.

The value of having stochastic versions of rearrangement inequalities, apart from purely mathematical interest, is manifested in their applicability in a surprisingly large number of statistical and reliability contexts such as, e.g., ranking problems, hypothesis testing, contamination models, and optimal assembly of systems.

In Section 2 we present some definitions and concepts. In Section 3 we establish preservation properties for the functions of interest under various statistical and mathematical operations. The theory of stochastically similarly ordered random vectors, which provides a unified approach to stochastic rearrangement inequalities, is developed in Section 4. In Section 4 we also show that many well known multivariate densities govern stochastically similarly ordered random vectors. In Section 5 we present some illustrative applications of the theory to ranking problems, hypothesis testing problems, and contamination models.

2. Preliminaries.

In this section we introduce definitions and preliminary results used in subsequent sections.

Let R^n denote Euclidean n-space. We define a partial ordering of $R^n \times R^n$ which, as MO (1979) show, unifies the study of deterministic rearrangement inequalities. To define this partial ordering we need some terminology and notation.

Let S_n denote the group of all permutations of $\{1, 2, \ldots, n\}$. An element of S_n is denoted by $\underset{\sim}{\pi} \equiv (\pi(1), \ldots, \pi(n))$. We sometimes identify S_n with the subset of R^n whose elements are those vectors with the integer components $1, 2, \ldots, n$. Let $\underset{\sim}{\pi}$ and $\underset{\sim}{\pi}'$ be elements of S_n. We say that $\underset{\sim}{\pi}'$ is a <u>simple transposition</u> of $\underset{\sim}{\pi}$ if there exist positive integers $1 \le i < j \le n$ such that $\pi(i) = \pi'(j) = \pi'(i) = \pi(j)$ and $\pi(k) = \pi'(k)$ for $k \ne i, j$. We write this as $\underset{\sim}{\pi} \overset{t_{ij}}{>} \underset{\sim}{\pi}'$. For $\underset{\sim}{\pi}, \underset{\sim}{\pi}' \in S_n$ we say that $\underset{\sim}{\pi}'$ is a <u>transposition</u> of $\underset{\sim}{\pi}$, written $\underset{\sim}{\pi} \overset{t}{\ge} \underset{\sim}{\pi}'$, if $\underset{\sim}{\pi} = \underset{\sim}{\pi}'$ or if $\underset{\sim}{\pi}'$ can be obtained from $\underset{\sim}{\pi}$ by a sequence of simple transpositions.

For a vector $\underset{\sim}{x} \in R^n$, we define $\underset{\sim}{x}\underset{\sim}{\pi}$ to be the vector $(x_{\pi(1)}, \ldots, x_{\pi(n)})$. We denote by $\vec{\underset{\sim}{x}}$ the vector obtained from $\underset{\sim}{x}$ by arranging the components of $\underset{\sim}{x}$ in increasing order. We say that $\underset{\sim}{x}'$ is a <u>transposition</u> of $\underset{\sim}{x}$ if $\underset{\sim}{x} = \vec{\underset{\sim}{x}}\underset{\sim}{\pi}$ and $\underset{\sim}{x}' = \vec{\underset{\sim}{x}}\underset{\sim}{\pi}'$, where $\underset{\sim}{\pi} \overset{t}{\ge} \underset{\sim}{\pi}'$. We write $\underset{\sim}{x} \overset{t}{\ge} \underset{\sim}{x}'$. We note that this defines a partial ordering of R^n. This partial ordering has been studied by Savage (1957), Lehmann (1966), and Hollander, Proschan, and Sethuraman (HPS) (1977), among others.

Let $(\underset{\sim}{x},\underset{\sim}{y}) \in R^n \times R^n$. The <u>orbit</u> of $(\underset{\sim}{x},\underset{\sim}{y})$ is the set $O_{\underset{\sim}{x},\underset{\sim}{y}} = \{(\underset{\sim}{x}\underset{\sim}{\pi}, \underset{\sim}{y}\underset{\sim}{\sigma}): \underset{\sim}{\pi}, \underset{\sim}{\sigma} \in S_n\}$. For a vector $\underset{\sim}{x} \in R^n$ the orbit of $\underset{\sim}{x}$ is defined similarly.

<u>2.1 Definition</u>. Let $(\underset{\sim}{x}, \underset{\sim}{y})$ and $(\underset{\sim}{x}',\underset{\sim}{y}')$ be two elements of $R^n \times R^n$ belonging to the same orbit. We say that $(\underset{\sim}{x},\underset{\sim}{y})$ is <u>more similarly arranged</u> than $(\underset{\sim}{x}',\underset{\sim}{y}')$ if there exist $\underset{\sim}{\pi}, \underset{\sim}{\sigma} \in S_n$ such that $\underset{\sim}{x}\underset{\sim}{\pi} = \underset{\sim}{x}'\underset{\sim}{\sigma} = \vec{\underset{\sim}{x}}$ and $\underset{\sim}{y}\underset{\sim}{\pi} \overset{t}{\ge} \underset{\sim}{y}'\underset{\sim}{\sigma}$. We write $(\underset{\sim}{x},\underset{\sim}{y}) \overset{a}{\ge} (\underset{\sim}{x}',\underset{\sim}{y}')$.

We refer to this partial ordering of $R^n \times R^n$ as the <u>arrangement ordering</u>. We write $(\underset{\sim}{x},\underset{\sim}{y}) \overset{a}{=} (\underset{\sim}{x}',\underset{\sim}{y}')$ if $(\underset{\sim}{x},\underset{\sim}{y}) \overset{a}{\ge} (\underset{\sim}{x}',\underset{\sim}{y}')$ and $(\underset{\sim}{x}',\underset{\sim}{y}') \overset{a}{\ge} (\underset{\sim}{x},\underset{\sim}{y})$. From the definition it is clear that once the arrangement ordering has been defined on the subset $\{(\vec{\underset{\sim}{x}}, \vec{\underset{\sim}{y}}\underset{\sim}{\pi}): \underset{\sim}{\pi} \in S_n\}$ of $O_{\underset{\sim}{x},\underset{\sim}{y}}$ it is completely determined for all of $O_{\underset{\sim}{x},\underset{\sim}{y}}$.

Figure 2.1 illustrates the arrangement ordering when $\vec{\underset{\sim}{x}} = (.5, 1, 3)$ and $\vec{\underset{\sim}{y}} = (2, 3.5, 4)$. An arrow in the diagram from an element $(\vec{\underset{\sim}{x}},\underset{\sim}{y})$ to an element $(\vec{\underset{\sim}{x}},\underset{\sim}{y}')$ means that $(\vec{\underset{\sim}{x}},\underset{\sim}{y}) \overset{a}{\ge} (\vec{\underset{\sim}{x}},\underset{\sim}{y}')$.

$$
\begin{tikzcd}[column sep=small]
& ((.5,\ 1,\ 3),\ (2,\ 3.5,\ 4)) \arrow[dl] \arrow[dr] & \\
((.5,\ 1,\ 3),\ (3.5,\ 2,\ 4)) \arrow[d] \arrow[dr, shift left=6] & & ((.5,\ 1,\ 3),\ (2,\ 4,\ 3.5)) \arrow[d] \arrow[dl, shift right=6] \\
((.5,\ 1,\ 3),\ (3.5,\ 4,\ 2)) \arrow[dr] & & ((.5,\ 1,\ 3),\ (4,\ 2,\ 3.5)) \arrow[dl] \\
& ((.5,\ 1,\ 3),\ (4,\ 3.5,\ 2)) &
\end{tikzcd}
$$

Figure 2.1. An Illustrative Arrangement Ordering.

<u>2.2 Remark</u>. Let $(\mathbf{x},\mathbf{y})$ denote the largest element of its orbit in the arrangement ordering, that is, $(\mathbf{x},\mathbf{y}) \overset{a}{\geq} (\mathbf{x}\boldsymbol{\pi},\mathbf{y}\boldsymbol{\sigma})$ for all $\boldsymbol{\pi}, \boldsymbol{\sigma} \in S_n$. Then it is easy to see that $(x_i - x_j)(y_i - y_j) \geq 0$ for all pairs i, j. In this case we say that $\mathbf{x}$ and $\mathbf{y}$ are <u>similarly arranged</u>. (HLP (1952) use the expression "similarly ordered".) We write $\mathbf{x} \overset{s}{=} \mathbf{y}$.

We next consider the classes of functions introduced by HPS (1977) which are order-preserving with respect to the transposition ordering and the arrangement ordering.

<u>2.3 Definition</u>. (i) A function f from R^n into R is said to be <u>decreasing in transposition</u> (DT) if $\mathbf{x} \overset{t}{\geq} \mathbf{x}'$ implies $f(\mathbf{x}) \geq f(\mathbf{x}')$ for all $\mathbf{x} \in R^n$. (ii) A function f from $R^n \times R^n$ into R is said to be <u>arrangement increasing</u> (AI) if $(\mathbf{x},\mathbf{y}) \overset{a}{\geq} (\mathbf{x}',\mathbf{y}')$ implies $f(\mathbf{x},\mathbf{y}) \geq f(\mathbf{x}',\mathbf{y}')$ for all $(\mathbf{x},\mathbf{y}) \in R^n \times R^n$.

Functions which are AI play an important role in the theory we develop. Their properties and many useful applications were developed by HPS (1977). They used the name "decreasing in transposition on $R^n \times R^n$". The present name is due to MO (1979).

The domain of an AI function is sometimes restricted to a subset $A \subset R^n \times R^n$. When this is done it is natural to require that A have the property that whenever

$(\underset{\sim}{x},\underset{\sim}{y}) \in A$, we have $(\underset{\sim}{x}\pi,\underset{\sim}{y}\sigma) \in A$ for all $\pi, \sigma \in S_n$. It is clear from Definition 2.3 that the AI property of functions is essentially a property of functions on $S_n \times S_n$. More precisely, $f(\underset{\sim}{x},\underset{\sim}{y})$ is AI on A if and only if for each fixed pair $(\vec{x},\vec{y}) \in A$, $f(\vec{x}\pi,\vec{y}\sigma)$ is AI as a function of π and σ. In the same way the DT property is a property of functions on S_n.

HPS (1977) give an alternative definition of an AI function. Proposition 2.4 below states that their definition is equivalent to Definition 2.3. The definition of HPS is more useful in determining whether or not a given function is AI.

<u>2.4 Proposition</u>. (MO, 1979). A function f from $R^n \times R^n$ into R is AI if and only if (i) $f(\underset{\sim}{x},\underset{\sim}{y}) = f(\underset{\sim}{x}\pi,\underset{\sim}{y}\pi)$ for $(\underset{\sim}{x},\underset{\sim}{y}) \in R^n \times R^n$, $\pi \in S_n$, and (ii) $f(\vec{x},\underset{\sim}{y}) \geq f(\vec{x},\underset{\sim}{y}')$, where $\underset{\sim}{y} \overset{t}{\geq} \underset{\sim}{y}'$.

<u>2.5 Remark</u>. A function satisfying condition (i) of Proposition 2.4 is called <u>permutation invariant</u> (PI). We also use this terminology to describe a function f defined on R^n satisfying $f(\underset{\sim}{x}\pi) = f(\underset{\sim}{x})$ for $\pi \in S_n$. Condition (ii) is equivalent to stating that for fixed $\vec{x}$, the function $f_{\vec{x}}(\underset{\sim}{y}) \equiv f(\vec{x},\underset{\sim}{y})$ is DT.

HPS (1977) give many examples of AI functions including a number of well-known densities in statistics.

<u>3. Positive Set Functions in Arrangement, Arrangement Preserving Kernels, and Their Preservation Properties</u>. In this section we define two new classes of functions and establish preservation properties of these functions under various statistical and mathematical operations.

<u>Subsection 3.1 Positive Set Functions in Arrangement and Arrangement Preserving Kernels</u>. Let f be a function of one or more vector arguments. Define the <u>difference operator</u> $\Delta^{ij}_{\underset{\sim}{x}_k}$ to be:

$$\Delta^{ij}_{\underset{\sim}{x}_k} f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k,\ldots,\underset{\sim}{x}_m) \equiv$$

$$f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k,\ldots,\underset{\sim}{x}_m) - f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k',\ldots,\underset{\sim}{x}_m)$$

where $\underset{\sim}{x}_k$ and $\underset{\sim}{x}_k'$ differ by a simple transposition of the i^{th} and the j^{th} components. We drop the superscript ij when it is understood.

3.1 Definition. A function f from $R^n \times R^n$ into R is called a positive set function in arrangement (PSA) if $\underset{\sim}{x} \overset{t_{ij}}{>} \underset{\sim}{x}'$ and $\underset{\sim}{y} \overset{t_{ij}}{\geq} \underset{\sim}{y}'$ for any pair $i < j$ implies $\Delta^{ij}_{\underset{\sim}{y}} \Delta^{ij}_{\underset{\sim}{x}} f(\underset{\sim}{x},\underset{\sim}{y}) \geq 0$, that is, $f(\underset{\sim}{x},\underset{\sim}{y}) - f(\underset{\sim}{x},\underset{\sim}{y}') - f(\underset{\sim}{x}',\underset{\sim}{y}) + f(\underset{\sim}{x}',\underset{\sim}{y}') \geq 0$. This is the condition needed on the joint density of $(\underset{\sim}{X},\underset{\sim}{Y})$ to obtain the stochastic version of the HLP inequality.

3.2 Proposition. The function f is PSA and PI iff f is AI. The proof is omitted because it is quite simple.

3.3 Definition. A function K from $(R^n \times R^n) \times (R^n \times R^n)$ into R is called an arrangement preserving (AP) kernel if:

(i) $K(\underset{\sim}{u}, \underset{\sim}{x}; \underset{\sim}{v}, \underset{\sim}{y})$ is PI in $(\underset{\sim}{u}, \underset{\sim}{x})$ and in $(\underset{\sim}{v}, \underset{\sim}{y})$, and

(ii) for all $\underset{\sim}{u}, \underset{\sim}{v} \in R^n$, $K(\vec{\underset{\sim}{u}}, \underset{\sim}{x}; \vec{\underset{\sim}{v}}, \underset{\sim}{y})$ is PSA in $(\underset{\sim}{x}, \underset{\sim}{y})$.

We show later in this section that under mild conditions on the measure m the function f given by $f(\underset{\sim}{u},\underset{\sim}{v}) \equiv \iint g(\underset{\sim}{x},\underset{\sim}{y})K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y})$ is AI whenever g is AI and K is AP. This preservation property of AP kernels is very useful in obtaining stochastic rearrangement inequalities for random vectors $(\underset{\sim}{X}(\underset{\sim}{u}), \underset{\sim}{Y}(\underset{\sim}{v}))$ which depend on parameters $(\underset{\sim}{u},\underset{\sim}{v})$.

It is clear from the definitions that the properties of these two classes of functions derive, as in the case of AI functions, from corresponding properties of functions on S_n. We make this more precise in the next remark.

3.4 Remark. (i) A function $f(\underset{\sim}{x},\underset{\sim}{y})$ is PSA (AI) in $(\underset{\sim}{x},\underset{\sim}{y})$ iff $f(\vec{\underset{\sim}{x}\pi},\vec{\underset{\sim}{y}\sigma})$ is PSA (AI) in $(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ for each fixed $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

(ii) A function $K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ is AP in $(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ iff $K(\underset{\sim}{u}\underset{\sim}{\rho},\underset{\sim}{x}\underset{\sim}{\pi};\ \underset{\sim}{v}\underset{\sim}{\tau},\underset{\sim}{y}\underset{\sim}{\sigma})$ is AP in $(\underset{\sim}{\rho},\underset{\sim}{\pi};\ \underset{\sim}{\tau},\underset{\sim}{\sigma})$ for each fixed $\underset{\sim}{u}$, $\underset{\sim}{x}$, $\underset{\sim}{v}$ and $\underset{\sim}{y}$.

In Theorem 3.5 below we give a method for constructing many examples of PSA functions and AP kernels using positive set functions. Recall that a <u>positive set function</u> is a nonnegative function ϕ defined on a subset $A \times B$ of R^2 satisfying $\phi(x_1,y_1) - \phi(x_1,y_2) - \phi(x_2,y_1) + \phi(x_2,y_2) \geq 0$ for all $x_1 < x_2$ in A and $y_1 < y_2$ in B.

<u>3.5 Theorem</u>. Let ϕ be a positive set function.

(i) If f_1 and f_2 are DT, then $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv \phi(f_1(\underset{\sim}{x}),f_2(\underset{\sim}{y}))$ is PSA.

(ii) If g_1 and g_2 are AI, then $K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}) \equiv \phi(g_1(\underset{\sim}{u},\underset{\sim}{x}),\ g_2(\underset{\sim}{v},\underset{\sim}{y}))$ is AP.

The proof is omitted because it is simple.

We conclude this subsection with some useful preservation properties of PSA functions and AP kernels. The proof of the next result is easy, and hence is omitted.

<u>3.6 Proposition</u>. Let $K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ be AP and let $f(\underset{\sim}{u},\underset{\sim}{v})$ be constant on the orbits $O_{\underset{\sim}{u},\underset{\sim}{v}}$ for each pair $\underset{\sim}{u},\underset{\sim}{v}$. Then $H(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}) \equiv K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})f(\underset{\sim}{u},\underset{\sim}{v})$ is AP.

HPS (1977) showed that if g and h are positive AI functions, then $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv g(\underset{\sim}{x},\underset{\sim}{y})h(\underset{\sim}{x},\underset{\sim}{y})$ is an AI function. Proposition 3.7 below is similar to this result.

<u>3.7 Proposition</u>. Let g be an AI function and let $h \geq 0$ be a PSA function. Then $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv g(\underset{\sim}{x},\underset{\sim}{y})h(\underset{\sim}{x},\underset{\sim}{y})$ is a PSA function.

<u>Proof</u>. Let $\underset{\sim}{x} \overset{t_{ij}}{\geq} \underset{\sim}{x}'$ and $\underset{\sim}{y} \overset{t_{ij}}{\geq} \underset{\sim}{y}'$. By definition,

$$\begin{aligned} f(\underset{\sim}{x},\underset{\sim}{y}) + f(\underset{\sim}{x}',\underset{\sim}{y}') &= g(\underset{\sim}{x},\underset{\sim}{y})h(\underset{\sim}{x},\underset{\sim}{y}) + g(\underset{\sim}{x}',\underset{\sim}{y}')h(\underset{\sim}{x}',\underset{\sim}{y}') \\ &= g(\underset{\sim}{x},\underset{\sim}{y})[h(\underset{\sim}{x},\underset{\sim}{y}) + h(\underset{\sim}{x}',\underset{\sim}{y}')] \quad \text{[since g is PI]} \\ &\geq g(\underset{\sim}{x},\underset{\sim}{y}')[h(\underset{\sim}{x}',\underset{\sim}{y}) + h(\underset{\sim}{x},\underset{\sim}{y}')] \quad \text{[since g is AI and h is PSA]} \\ &= f(\underset{\sim}{x}',\underset{\sim}{y}) + f(\underset{\sim}{x},\underset{\sim}{y}'). \quad \text{[since g is PI].} \end{aligned}$$

Thus f is PSA. ||

3.8 Corollary. (i) Let g_1 and g_2 be DT and let h be AI. Then $f(\underset{\sim}{x},\underset{\sim}{y}) = g_1(\underset{\sim}{x})h(\underset{\sim}{x},\underset{\sim}{y})g_2(\underset{\sim}{y})$ is PSA.

(ii) Let g_1 and g_2 be PI on R^n and let h be PSA. Then $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv g_1(\underset{\sim}{x})h(\underset{\sim}{x},\underset{\sim}{y})g_2(\underset{\sim}{y})$ is PSA.

Proof. Part (i) follows from Proposition 3.7 and Proposition 3.5 (i). Part (ii) follows from Proposition 3.7 and the fact that if g_1 and g_2 are PI functions on R^n, then $g(\underset{\sim}{x},\underset{\sim}{y}) \equiv g_1(\underset{\sim}{x})g_2(\underset{\sim}{y})$ is AI. ||

Subsection 3.2 Preservation Properties of Positive Set Functions in Arrangement and Arrangement Preserving Kernels. Next we present the main preservation properties of PSA functions and AP kernels.

Let m denote a measure on the Borel σ-field of $R^n \times R^n$ such that for all permutations $\underset{\sim}{\pi}$ and all measurable subsets $A \times B$ of $R^n \times R^n$ we have $m(A \times B) = m\{(\underset{\sim}{x}\underset{\sim}{\pi},\underset{\sim}{y}\underset{\sim}{\pi}): (\underset{\sim}{x},\underset{\sim}{y}) \in A \times B\}$. We call such a measure permutation invariant. We also call a measure m on the Borel σ-field of R^n permutation invariant if for all measurable subsets A of R^n and all $\underset{\sim}{\pi} \in S_n$, $m(A) = m\{\underset{\sim}{x}\underset{\sim}{\pi}: \underset{\sim}{x} \in A\}$.

3.9 Theorem. Let g be PSA and let K be AP. Then assuming the integral exists finitely, $f(u,v) \equiv \iint g(\underset{\sim}{x},\underset{\sim}{y})K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y})$ is PSA.

Proof. Let $\Delta_{\underset{\sim}{u}} = \Delta_{\underset{\sim}{u}}^{ij}$ and $\Delta_{\underset{\sim}{v}} = \Delta_{\underset{\sim}{v}}^{ij}$. We have $\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}f(\underset{\sim}{u},\underset{\sim}{v}) = \iint g(\underset{\sim}{x},\underset{\sim}{y})\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y})$.

Partition $R^n \times R^n$ into the four subsets $\{(\underset{\sim}{x},\underset{\sim}{y}): x_i \le x_j,\ y_i \le y_j\}$, $\{(x,y): x_i > x_j,\ y_i \le y_j\}$, $\{(\underset{\sim}{x},\underset{\sim}{y}): x_i \le x_j,\ y_i > y_j\}$, and $\{(x,y): x_i > x_j,\ y_i > y_j\}$. Consider the integral

$$(3.1)\quad \int\limits_{x_i \le x_j,\ y_i \le y_j}\int \{g(\underset{\sim}{x},\underset{\sim}{y})\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}) + g(\underset{\sim}{x}',\underset{\sim}{y})\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x}';\ \underset{\sim}{v},\underset{\sim}{y})$$

$$+ g(\underset{\sim}{x},\underset{\sim}{y}')\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}') + g(\underset{\sim}{x}',\underset{\sim}{y}')\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x}';\ \underset{\sim}{v},\underset{\sim}{y}')\}m(d\underset{\sim}{x},d\underset{\sim}{y}).$$

By the permutation invariance of K in the first and second pairs of arguments we may rewrite the above expression as

$$(3.2)\quad \int_{x_i \le x_j,\ y_i \le y_j} \int \{g(\underset{\sim}{x},\underset{\sim}{y})\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}) - g(\underset{\sim}{x}',\underset{\sim}{y})\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$$

$$- g(\underset{\sim}{x},\underset{\sim}{y}')\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y}) + g(\underset{\sim}{x}',\underset{\sim}{y}')\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})\}m(d\underset{\sim}{x},d\underset{\sim}{y}).$$

We notice that for $x_i = x_j$ or $y_i = y_j$, the integrand is zero. Hence the integral in (3.2) is equal to the integral in (3.1).

The integral in (3.2) is equal to

$$\int_{x_i \le x_j,\ y_i \le y_j} \int [\Delta_{\underset{\sim}{x}}\Delta_{\underset{\sim}{y}}g(\underset{\sim}{x},\underset{\sim}{y})][\Delta_{\underset{\sim}{u}}\Delta_{\underset{\sim}{v}}K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})]m(d\underset{\sim}{x},d\underset{\sim}{y}).$$

Since each expression in brackets is nonnegative by hypothesis, the integral is nonnegative. Thus f is PSA. ||

In Proposition 3.2 we proved that a PSA function which is permutation invariant is AI. This result together with Theorem 3.9 yields a preservation theorem for AI functions which we prove next.

<u>3.10 Theorem</u>. Let g be AI and let K be AP. Then assuming the integral exists finitely, $f(\underset{\sim}{u},\underset{\sim}{v}) \equiv \iint g(\underset{\sim}{x},\underset{\sim}{y})K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y})$ is AI.

<u>Proof</u>. By Theorem 3.9, K is PSA. Hence by Proposition 3.2 it suffices to show that K is PI. Let $\underset{\sim}{\pi} \in S_n$ and let $\underset{\sim}{\pi}^{-1}$ be its inverse. Then

$$\begin{aligned} f(\underset{\sim}{u}\underset{\sim}{\pi},\underset{\sim}{v}\underset{\sim}{\pi}) &= \iint g(\underset{\sim}{x},\underset{\sim}{y})K(\underset{\sim}{u}\underset{\sim}{\pi},\underset{\sim}{x};\ \underset{\sim}{v}\underset{\sim}{\pi},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y}) \\ &= \iint g(\underset{\sim}{x}\underset{\sim}{\pi}^{-1},\underset{\sim}{y}\underset{\sim}{\pi}^{-1})K(\underset{\sim}{u},\underset{\sim}{x}\underset{\sim}{\pi}^{-1};\ \underset{\sim}{v},\underset{\sim}{y}\underset{\sim}{\pi}^{-1})m(d\underset{\sim}{x},d\underset{\sim}{y}) \\ &= \iint g(\underset{\sim}{x},\underset{\sim}{y})K(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y}) = f(\underset{\sim}{u},\underset{\sim}{v}) \end{aligned}$$

by the permutation invariance properties of g and k and a change of variables. ||

Theorem 3.10 is the main result of this section. It allows us to obtain stochastic versions of rearrangement inequalities.

Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be a pair of nonnegative random vectors with a joint density which is PSA. For $\underset{\sim}{\pi} \in S_n$ let $\underset{\sim}{X}_{\underset{\sim}{\pi}}$ denote the vector $(X_{\pi(1)},\ldots,X_{\pi(n)})$. We shall see in the next section that the joint density of $(\underset{\sim}{X}_{\underset{\sim}{\pi}},\underset{\sim}{Y}_{\underset{\sim}{\sigma}})$ is an AP function of

$(\underset{\sim}{\pi},\underset{\sim}{x};\ \underset{\sim}{g},\underset{\sim}{y})$. Let K in Theorem 3.10 be the joint density of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$. Then the theorem states that if g is AI in $(\underset{\sim}{x},\underset{\sim}{y})$, then $Eg(\underset{\sim}{X},\underset{\sim}{Y})$ is AI in the indices of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$. Another way of stating this is to say that for all $\underset{\sim}{\pi} \in S_n$

$$g(X_1,\ldots,X_n;\ Y_1,\ldots,Y_n) \overset{st.}{\geq} g(X_1,\ldots,X_n;\ Y_{\pi(1)},\ldots Y_{\pi(n)})$$

$$\overset{st.}{\geq} g(X_1,\ldots,X_n;\ Y_n,\ldots,Y_1).$$

Since $g(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=1}^{n} x_i y_i$ is AI, we see that Theorem 3.10 yields a stochastic version of the inequality of HLP. We will show in Section 5 that we can obtain stochastic versions of many other deterministic rearrangement inequalities for a larger class of random vectors $(\underset{\sim}{X},\underset{\sim}{Y})$ which contains those having PSA densities. In that section we will also illustrate the usefulness of stochastic rearrangement inequalities in statistical applications.

As a consequence of Theorem 3.9, we have proved a composition theorem for AP kernels. This result, presented in Theorem 3.11, is important in that it provides a method for generating new AP kernels from known ones.

<u>3.11 Theorem</u>. Let G and H be AP kernels. Then

$$K(\underset{\sim}{u},\underset{\sim}{w};\ \underset{\sim}{v},\underset{\sim}{z}) = \iint G(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})H(\underset{\sim}{x},\underset{\sim}{w};\ \underset{\sim}{y},\underset{\sim}{z})m(d\underset{\sim}{x},d\underset{\sim}{y}) \text{ is AP.}$$

The proof is straightforward and hence is omitted.

We next present another important preservation result which enlarges each of the classes of functions we are studying. It states that each of the classes of AI functions, PSA functions, and AP kernels is closed under mixtures.

<u>3.12 Theorem</u>. Let (Θ,F,μ) be a measure space and let $F_\theta(\underset{\sim}{x})$ be measurable as a function of $\theta \in \Theta$ for each $\underset{\sim}{x}$. Define $F \equiv \int F_\theta\ \mu(d\theta)$. Then it follows that (i) if F_θ is AI for each θ, then F is AI; (ii) if F_θ is PSA for each θ, then F is PSA; (iii) if F_θ is AP for each θ, then F is AP.

The proof of this result is straightforward and is omitted. Part (i) of Theorem 3.12 is due to HPS (1977).

HPS (1977) prove as their main result a preservation theorem for AI functions. We show in Theorem 3.13 below that this preservation theorem is a special case of Theorem 3.10. Thus the theory we have developed extends the theory of AI functions developed by HPS (1977).

3.13 Theorem. (HPS,1977). Let g and h be AI functions. Then

$$f(\underset{\sim}{u},\underset{\sim}{v}) = \int g(\underset{\sim}{u},\underset{\sim}{x})h(\underset{\sim}{x},\underset{\sim}{v})m(d\underset{\sim}{x}) \quad \text{is AI.}$$

The proof is straightforward and is omitted.

In Theorem 3.14 below we present a generalization of the preservation theorem of HPS (1977) for AI functions presented in Theorem 3.13.

3.14 Theorem. Let g be PSA, h be AI, and m be PI. Then

$$f(\underset{\sim}{x},\underset{\sim}{y}) = \int g(\underset{\sim}{x},\underset{\sim}{z})h(\underset{\sim}{z},\underset{\sim}{y})\, m(d\underset{\sim}{z}) \quad \text{is PSA.}$$

Proof. Let $\Delta_{\underset{\sim}{x}} = \Delta^{ij}_{\underset{\sim}{x}}$ and $\Delta_{\underset{\sim}{y}} = \Delta^{ij}_{\underset{\sim}{y}}$. By definition,

$$\Delta_{\underset{\sim}{x}}\Delta_{y}\, f(\underset{\sim}{x},\underset{\sim}{y}) = \int \Delta_{\underset{\sim}{x}}\Delta_{\underset{\sim}{y}}\, g(\underset{\sim}{x},\underset{\sim}{z})h(\underset{\sim}{z},\underset{\sim}{y})m(d\underset{\sim}{z}) =$$

$$\int \{[g(\underset{\sim}{x},\underset{\sim}{z}) - g(\underset{\sim}{x}',\underset{\sim}{z})]h(\underset{\sim}{z},\underset{\sim}{y}) + [g(\underset{\sim}{x}',\underset{\sim}{z}) - g(\underset{\sim}{x},\underset{\sim}{z})]h(\underset{\sim}{z},\underset{\sim}{y}')\}m(d\underset{\sim}{z}).$$

By partitioning R^n into the sets $\{\underset{\sim}{z}: z_i \le z_j\}$ and $\{\underset{\sim}{z}: z_i > z_j\}$, we may write the above expression as

$$\int_{z_i \le z_j} [g(\underset{\sim}{x},\underset{\sim}{z}) - g(\underset{\sim}{x}',\underset{\sim}{z})][h(\underset{\sim}{z},\underset{\sim}{y}) - h(\underset{\sim}{z},\underset{\sim}{y}')] + [g(\underset{\sim}{x},\underset{\sim}{z}') - g(\underset{\sim}{x}',\underset{\sim}{z}')][h(\underset{\sim}{z}',\underset{\sim}{y}) - h(\underset{\sim}{z}',\underset{\sim}{y}')]m(d\underset{\sim}{z})$$

$$= \int_{z_i \le z_j} [g(\underset{\sim}{x},\underset{\sim}{z}) + g(\underset{\sim}{x}',\underset{\sim}{z}') - g(\underset{\sim}{x}',\underset{\sim}{z}) - g(\underset{\sim}{x},\underset{\sim}{z}')][h(\underset{\sim}{z},\underset{\sim}{y}) - h(\underset{\sim}{z},\underset{\sim}{y}')]m(d\underset{\sim}{z})$$

by the permutation invariance of h. The last integral is nonnegative since each expression in brackets is nonnegative by hypothesis.

Thus f is PSA. ||

HPS (1977) applied their preservation theorem to ranking problems in nonparametric statistics. In Section 4 we show how to obtain some of their results under slightly more general assumptions. In addition, we obtain new results applicable to other ranking problems.

Theorem 3.16 below and its corollaries will be useful in these applications to ranking problems. In it we show how to obtain new PSA and AI functions by composing these functions with certain transformations on R^n. By a transformation we mean a mapping from R^n into R^n of the form $\underset{\sim}{t}(\underset{\sim}{x}) = (t_1(\underset{\sim}{x}),\ldots,t_n(\underset{\sim}{x}))$, where t_i is a function on R^n. We write $-\underset{\sim}{x}$ for the vector $(-x_1,-x_2,\ldots,-x_n)$.

3.15 Definition. Let $\underset{\sim}{t}$ be a transformation from R^n into R^n. Then t is called a rank-like (reverse rank-like) transformation if it satisfies:

(i) $\underset{\sim}{t}(\underset{\sim}{x}\underset{\sim}{\pi}) = \underset{\sim}{t}(\underset{\sim}{x})\underset{\sim}{\pi}$ for all $\underset{\sim}{\pi} \in S_n$, and
(ii) $\underset{\sim}{t}(\underset{\sim}{x})$ and $\underset{\sim}{x}(-\underset{\sim}{x})$ are similarly arranged.

One example of a rank-like transformation (which motivates the name "rank-like") is the rank order transformation, important in nonparametric statistics. The rank order transformation is the transformation $\underset{\sim}{r}(\underset{\sim}{x}) \equiv (r_1(\underset{\sim}{x}),\ldots,r_n(\underset{\sim}{x}))$, where $r_i(\underset{\sim}{x})$ is the rank of x_i among $x_1,\ldots,x_n$; in the case of tied x's, average ranks are used. The vector $\underset{\sim}{r}(\underset{\sim}{x})$ is called the rank order of $\underset{\sim}{x}$. Another example is the transformation $\underset{\sim}{f}(\underset{\sim}{x}) = (f(x_1),\ldots,f(x_n))$, where f is an increasing function. If f is decreasing then $f(\underset{\sim}{x})$ is a reverse rank-like transformation.

We now obtain some results concerning rank-like transformations.

3.16 Theorem. Let f be a PSA (AI) function. Let $\underset{\sim}{t}$ and $\underset{\sim}{s}$ be either both rank-like or both reverse rank-like transformations on R^n. Then $\tilde{f}(\underset{\sim}{x},\underset{\sim}{y}) \equiv f(\underset{\sim}{t}(\underset{\sim}{x}),\underset{\sim}{s}(\underset{\sim}{y}))$ is PSA(AI).

Proof. Assume that f is PSA. Let $\underset{\sim}{x} \overset{t_{ij}}{\geq} \underset{\sim}{x}'$ and $\underset{\sim}{y} \overset{t_{ij}}{\geq} \underset{\sim}{y}'$. We have

$$\tilde{f}(\underset{\sim}{x},\underset{\sim}{y}) + f(\underset{\sim}{x}',\underset{\sim}{y}') = f(\underset{\sim}{t}(\underset{\sim}{x}), \underset{\sim}{s}(\underset{\sim}{y})) + f(\underset{\sim}{t}(\underset{\sim}{x}'), \underset{\sim}{s}(\underset{\sim}{y}')) =$$

$$f(\underset{\sim}{t}(\underset{\sim}{x}),\underset{\sim}{s}(\underset{\sim}{y})) + f(\underset{\sim}{t}'(\underset{\sim}{x}), \underset{\sim}{s}'(\underset{\sim}{y})) \geq \text{[by (i) of Definition 3.15]}$$

$$f(\underset{\sim}{t}'(\underset{\sim}{x}), \underset{\sim}{s}(\underset{\sim}{y})) + f(\underset{\sim}{t}(\underset{\sim}{x}), \underset{\sim}{s}'(\underset{\sim}{y})) = \text{[by (ii) of Definition 3.15]}$$

$$f(\underset{\sim}{t}(\underset{\sim}{x}'), \underset{\sim}{s}(\underset{\sim}{y})) + f(\underset{\sim}{t}(\underset{\sim}{x}), \underset{\sim}{s}(\underset{\sim}{y}')) = \tilde{f}(\underset{\sim}{x}',\underset{\sim}{y}) + \tilde{f}(\underset{\sim}{x},\underset{\sim}{y}').$$

Thus, $\tilde{f}$ is PSA.

Now assume that f is AI. By Proposition 3.2 f is PSA. Hence, by what we have shown above, $\tilde{f}$ is PSA. To show that $\tilde{f}$ is AI it remains to show, by Proposition 3.2, that $\tilde{f}$ is PI . Let $\underset{\sim}{\pi} \in S_n$. Then for all $\underset{\sim}{x}$, $\underset{\sim}{y} \in R^n$ we have

$$\tilde{f}(\underset{\sim}{x\pi},\underset{\sim}{y\pi}) = f(\underset{\sim}{t}(\underset{\sim}{x\pi}), \underset{\sim}{s}(\underset{\sim}{y\pi})) = f(\underset{\sim}{t}(\underset{\sim}{x})\underset{\sim}{\pi}, \underset{\sim}{s}(\underset{\sim}{y})\underset{\sim}{\pi}) = f(\underset{\sim}{t}(\underset{\sim}{x}), \underset{\sim}{s}(\underset{\sim}{y})) = \tilde{f}(\underset{\sim}{x},\underset{\sim}{y}).$$

Thus $\tilde{f}$ is AI. ||

Using the examples of ranklike transformations mentioned previously, we can obtain results concerning rank orders and increasing (decreasing) transformations.

3.17 Corollary. Let f be a PSA(AI) function. (i) If $\underset{\sim}{r}(\underset{\sim}{x})$ and $\underset{\sim}{s}(\underset{\sim}{y})$ are the rank order transformations of $\underset{\sim}{x}$ and $\underset{\sim}{y}$, then $f(\underset{\sim}{r}(\underset{\sim}{x}), \underset{\sim}{s}(\underset{\sim}{y}))$ is PSA(AI). (ii) If g and h are either both increasing or both decreasing functions, then $f(\underset{\sim}{g}(\underset{\sim}{x}), \underset{\sim}{h}(\underset{\sim}{y})) \equiv$ $f((g(x_1), g(x_2), \ldots, g(x_n)), (h(y_1), h(y_2), \ldots, h(y_n)))$ is PSA(AI).

Since the transformation $\underset{\sim}{t}(\underset{\sim}{x}) \equiv \underset{\sim}{x}$ is clearly rank-like, we obtain immediately:

3.18 Corollary. Let $\underset{\sim}{t}$ be a rank-like transformation and let $f(\underset{\sim}{x},\underset{\sim}{y})$ be PSA(AI). Then $f(\underset{\sim}{x},\underset{\sim}{t}(\underset{\sim}{y}))$ is PSA(AI).

Another useful corollary of Theorem 3.16 is:

3.19 Corollary. Let $\underset{\sim}{t}$ and $\underset{\sim}{s}$ be either both rank-like or both reverse rank-like and let $K(\underset{\sim}{u},\underset{\sim}{x}; \underset{\sim}{v},\underset{\sim}{y})$ be AP. Then $K(\underset{\sim}{u},\underset{\sim}{t}(\underset{\sim}{x}); \underset{\sim}{v},\underset{\sim}{s}(\underset{\sim}{y}))$ is AP.

4. Stochastically Similarly Arranged Pairs of Random Vectors. In the first subsection of this section we introduce the notion of stochastically similarly arranged pairs of random vectors.

We prove that if $(\underset{\sim}{X},\underset{\sim}{Y})$ are stochastically similarly arranged (SSA) then $f(\underset{\sim}{X},\underset{\sim}{Y}) \overset{st.}{\geq} f(\underset{\sim}{X}_{\underset{\sim}{\pi}},\underset{\sim}{Y}_{\underset{\sim}{\sigma}})$ for all AI functions f, where $(\underset{\sim}{X}_{\underset{\sim}{\pi}},\underset{\sim}{Y}_{\underset{\sim}{\sigma}})$ is the random vector $(X_{\pi(1)},\ldots,X_{\pi(n)};\ Y_{\sigma(1)},\ldots,Y_{\sigma(n)})$. We show that if $(\underset{\sim}{X},\underset{\sim}{Y})$ have a joint density which is PSA or AP then $(\underset{\sim}{X},\underset{\sim}{Y})$ are stochastically similarly arranged. In the case where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ depend on parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$, we define the slightly more general notion of $(\underset{\sim}{X},\underset{\sim}{Y})$ being stochastically arranged (SA) like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. We prove results analogous to the above for these pairs of random vectors.

In Subsection 4.2 we show that many multivariate densities of interest in statistical practice govern pairs of random vectors which are SSA or SA like parameters $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ by showing that they belong to the class of PSA functions or AP kernels.

We show in Subsection 4.3 that under certain statistical operations on pairs of SSA random vectors, the property of being SSA is preserved. (As an example, we show that this property is preserved under mixtures.)

We conclude the section with some generalizations of the work of HPS (1977).

Subsection 4.1 A Stochastic Arrangement Ordering for Pairs of Random Vectors. We begin this section with the definition of a new notion of stochastic comparison between pairs of random vectors. This notion leads to a stochastic version of the arrangement ordering for pairs of n-tuples presented in Definition 2.1 .

4.1 Definition. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ and $(\underset{\sim}{W},\underset{\sim}{Z})$ be pairs of random vectors. We say that $(\underset{\sim}{X},\underset{\sim}{Y})$ are stochastically more similarly arranged than $(\underset{\sim}{W},\underset{\sim}{Z})$ if, for each AI function f we have $Ef(\underset{\sim}{X},\underset{\sim}{Y}) \geq Ef(\underset{\sim}{W},\underset{\sim}{Z})$, or, equivalently, $f(\underset{\sim}{X},\underset{\sim}{Y}) \overset{st.}{\geq} f(\underset{\sim}{W},\underset{\sim}{Z})$. We write this as $(\underset{\sim}{X},\underset{\sim}{Y}) \overset{st.a.}{\geq} (\underset{\sim}{W},\underset{\sim}{Z})$.

We frequently consider pairs of random vectors $(X(\alpha),Y(\beta)) \equiv (X_1(\alpha_1),\ldots,X_n(\alpha_n); Y_1(\beta_1,\ldots,Y_n(\beta_n))$ with associated vector parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$. As noted above, we write $(\underset{\sim}{X}_{\underset{\sim}{\pi}},\underset{\sim}{Y}_{\underset{\sim}{\sigma}})$ for $(X_{\pi(1)},\ldots X_{\pi(n)}; Y_{\sigma(1)},\ldots,Y_{\sigma(n)})$. In this case we can think of the indices π and σ as being the parameters of the random vectors $\underset{\sim}{X}_{\pi}$ and $\underset{\sim}{Y}_{\sigma}$. In Theorem 4.2 below we impose a condition on the joint distribution of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ so that $(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \overset{a}{\geq} (\underset{\sim}{\alpha}',\underset{\sim}{\beta}')$ implies that $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\underset{\sim}{Y}(\underset{\sim}{\beta})) \overset{st.}{\geq} (\underset{\sim}{X}(\underset{\sim}{\alpha}'),\underset{\sim}{Y}(\underset{\sim}{\beta}'))$. Thus the arrangement ordering of the parameters is reflected in a stochastic arrangement ordering of the random vectors.

We use the notation $\vec{\underset{\sim}{X}}$ for the vector of order statistics $(X_{(1)},\ldots,X_{(n)})$.

<u>4.2 Theorem</u>. *Let $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\underset{\sim}{Y}(\underset{\sim}{\beta}))$ be a pair of random vectors with parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$. Let*

$$(4.1)\qquad P(\underset{\sim}{X}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}\underset{\sim}{\pi},\ \underset{\sim}{Y}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}}\underset{\sim}{\sigma} \,|\, \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}})$$

be a version of the conditional probability which is an AP function of $(\underset{\sim}{\alpha},\underset{\sim}{\pi};\ \underset{\sim}{\beta},\underset{\sim}{\sigma})$ for each fixed pair $(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}})$. Then $(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \overset{a}{\geq} (\underset{\sim}{\alpha}',\underset{\sim}{\beta}')$ implies $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\underset{\sim}{Y}(\underset{\sim}{\beta})) \overset{st.a.}{\geq} (\underset{\sim}{X}(\underset{\sim}{\alpha}'),\underset{\sim}{Y}(\underset{\sim}{\beta}'))$.

<u>Proof</u>. The result is a direct consequence of Theorem 3.10. ||

In the special case where we interpret the indices of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ as parameters, we have the following corollary of Theorem 4.2 .

<u>4.3 Corollary</u>. For a pair of random vectors $(\underset{\sim}{X},\underset{\sim}{Y})$, let

$$(4.2)\qquad P(\underset{\sim}{X} = \vec{\underset{\sim}{x}}\underset{\sim}{\pi},\ \underset{\sim}{Y} = \vec{\underset{\sim}{y}}\underset{\sim}{\sigma} \,|\, \vec{\underset{\sim}{X}} = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}} = \vec{\underset{\sim}{y}})$$

be a version of the conditional probability which is a PSA function of $(\underset{\sim}{\pi},\underset{\sim}{\sigma})$ for each fixed pair $(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}})$. Then $(\underset{\sim}{\rho},\underset{\sim}{\tau}) \overset{a}{\geq} (\underset{\sim}{\rho}',\underset{\sim}{\tau}')$ implies $(\underset{\sim}{X}_{\underset{\sim}{\rho}},\underset{\sim}{Y}_{\underset{\sim}{\tau}}) \overset{st.a.}{\geq} (\underset{\sim}{X}_{\underset{\sim}{\rho}'},\underset{\sim}{Y}_{\underset{\sim}{\tau}'})$.

Theorem 4.2 and Corollary 4.3 motivate the next two definitions.

<u>4.4 Definition</u>. We say that $(\underset{\sim}{X}(\alpha),\ \underset{\sim}{Y}(\beta))$ are <u>stochastically arranged like</u> $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$) if the random vectors $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\underset{\sim}{Y}(\underset{\sim}{\beta}))$ are such that a version of the conditional probability in (4.1) is AP.

We will sometimes drop the parameters $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ and write $(\underset{\sim}{X}, \underset{\sim}{Y})$ for $(\underset{\sim}{X}(\underset{\sim}{\alpha}), \underset{\sim}{Y}(\underset{\sim}{\beta}))$. If F is a c.d.f. governing a pair of random vectors which are SA like $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$ we say that F is SA like $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$.

<u>4.5 Definition</u>. We say that $(\underset{\sim}{X}, \underset{\sim}{Y})$ are <u>stochastically similarly arranged</u> (SSA) if the random vectors $(\underset{\sim}{X}, \underset{\sim}{Y})$ are such that a version of the conditional probability in (4.2) is PSA.

We say that the c.d.f. governing a pair of SSA random vectors is SSA. It can be shown that if the pair of random vectors $(\underset{\sim}{X}, \underset{\sim}{Y})$ is degenerate at $(\underset{\sim}{x}, \underset{\sim}{y})$, then $(\underset{\sim}{X}, \underset{\sim}{Y})$ are SSA iff $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are similarly arranged. Hence the notion of SSA random vectors is a stochastic generalization of the deterministic notion of similarly arranged pairs of vectors.

<u>4.6 Remark</u>. Note that if $(\underset{\sim}{X}(\underset{\sim}{\alpha}), \underset{\sim}{Y}(\underset{\sim}{\beta}))$ are SA like $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$, then for $(\vec{\underset{\sim}{\alpha}}, \vec{\underset{\sim}{\beta}})$ fixed, $(\underset{\sim}{X}(\vec{\underset{\sim}{\alpha}}), \underset{\sim}{Y}(\vec{\underset{\sim}{\beta}}))$ are SSA. Conversely, if $(\underset{\sim}{X}, \underset{\sim}{Y})$ are SSA, then $(\underset{\sim}{X}_{\underset{\sim}{\rho}}, \underset{\sim}{Y}_{\underset{\sim}{\tau}})$ are SA like $(\underset{\sim}{\rho}, \underset{\sim}{\tau})$.

In Section 5 we use Theorem 4.2 and Corollary 4.3 to obtain numerous stochastic rearrangement inequalities for pairs of random vectors which are SA like (α, β) or SSA. We show that these inequalities contain as special cases their deterministic counterparts.

In the next theorem we present a sufficient condition for a pair of random vectors $(\underset{\sim}{X}, \underset{\sim}{Y})$ to be SSA.

<u>4.7 Theorem</u>. Let $(\underset{\sim}{X}, \underset{\sim}{Y})$ have a PSA density with respect to a permutation invariant measure. Then $(\underset{\sim}{X}, \underset{\sim}{Y})$ are SSA.

The proof is straightforward and is omitted.

Similarly, we have a sufficient condition for $(\underset{\sim}{X}(\underset{\sim}{\alpha}), \underset{\sim}{Y}(\underset{\sim}{\beta}))$ to be SA like $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$ which we state next.

<u>4.8 Theorem</u>. Let $(\underset{\sim}{X}(\underset{\sim}{\alpha}), \underset{\sim}{Y}(\underset{\sim}{\beta}))$ have an AP density with respect to a permutation invariant measure. Then $(\underset{\sim}{X}(\underset{\sim}{\alpha}), \underset{\sim}{Y}(\underset{\sim}{\beta}))$ are SA like $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$.

The proof is straightforward and hence is omitted.

In the next theorem we give a sufficient condition for the distribution function of a pair of random vectors to be PSA.

4.9 Theorem. Let $f(\underset{\sim}{x},\underset{\sim}{y})$ be a PSA density and let $F(\underset{\sim}{x},\underset{\sim}{y})$ be its corresponding distribution function. Then $F(\underset{\sim}{x},\underset{\sim}{y})$ is PSA.

Proof. Write

$$F(\underset{\sim}{x},\underset{\sim}{y}) = \int\int \prod_{i=1}^{n} I_{(o,x_i)}(u_i) \prod_{i=1}^{n} I_{(o,y_i)}(v_i) f(\underset{\sim}{u},\underset{\sim}{v}) m(d\underset{\sim}{u},d\underset{\sim}{v}).$$

Since each product of indicator functions is AI, $\prod_{i=1}^{n} I_{(o,x_i)}(u_i) \times \prod_{i=1}^{n} I_{(o,y_i)}(v_i)$ is AP by Theorem 3.5 (ii). Since f is PSA, F is PSA by Theorem 3.9.

An analogous result holds when $(\underset{\sim}{X},\underset{\sim}{Y})$ has an AP density.

4.10 Theorem. Let $f(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ be an AP density and let $F(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ be its corresponding distribution function. Then $F(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$ is AP.

The proof of Theorem 4.10 follows from Theorem 3.11 in a manner similar to that of Theorem 4.9, and hence, is omitted.

The results in this subsection allow us to obtain stochastic rearrangement inequalities for a large class of random vectors which contains those pairs $(\underset{\sim}{X},\underset{\sim}{Y})$ having PSA or AP densities. Many examples of such densities are given in the next subsection.

Subsection 4.2 Densities Governing Random Vectors Stochastically Arranged like Parameters $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. The purpose of this subsection is to show that many multivariate densities of interest in statistical practice govern pairs of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. In the next subsection we show how operations of statistical interest on pairs of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ preserve this property. Hence, from the basic examples given in this subsection, many other examples can be constructed.

In Theorem 4.11 below we show how to construct pairs of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ from independent random vectors with AI densities. Since a large number of well-known densities are AI, this result provides many examples of pairs of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. By Remark 4.6, corresponding to each one of these examples we can obtain an example of a pair of random vectors which are SSA.

<u>4.11 Theorem</u>. Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be independent random vectors in $R^n \times R^n$ each having an AI density. Then the joint density of $(\underset{\sim}{X},\underset{\sim}{Y})$ is AP.

<u>Proof</u>. The result follows immediately from Theorem 3.5 (ii) with $\phi(x,y) = xy$. ||

The following examples of AI densities can be used to construct AP densities. (See HPS, 1977 for proofs).

<u>4.12a Multinomial</u>: $g_1(\underset{\sim}{u},\underset{\sim}{x}) = N! \prod_{i=1}^{n} \frac{u_i^{x_i}}{x_i!}$, where

$$0 < u_i < 1,\ x_i = 0,1,2,\ldots,i = 1,\ldots,n,\ \sum_{i=1}^{n} u_i = 1, \text{ and } \sum_{i=1}^{n} x_i = N.$$

<u>4.12b Negative multinomial</u>: $g_2(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\Gamma\left(N + \sum_{i=1}^{n} x_i\right)}{\Gamma(N)} \left(1 + \sum_{i=1}^{n} u_i\right)^{-N - \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{u_i^{x_i}}{x_i!},$$

where $u_i > 0$, $x_i = 0,1,\ldots$, $i = 1, \ldots, n$, and $N > 0$.

<u>4.12c Multivariate hypergeometric</u>: $g_3(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\prod_{i=1}^{n} \begin{bmatrix} u_i \\ x_i \end{bmatrix} \Big/ \begin{bmatrix} \sum_{i=1}^{n} u_i \\ N \end{bmatrix}, \text{ where } u_i > 0,\ x_i = 0,1,\ldots,$$

$$\sum_{i=1}^{n} x_i = N < \sum_{i=1}^{n} u_i.$$

<u>4.12d Dirichlet</u>: $g_4(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\Gamma\left(\theta + \sum_{i=1}^{n} u_i\right)}{\Gamma(\theta) \prod_{i=1}^{n} \Gamma(u_i)} \left(1 - \sum_{i=1}^{n} x_i\right)^{\theta-1} \prod_{i=1}^{n} x_i^{u_i-1},$$

where $u_i > 0$, $x_i \geq 0$, $i=1, \ldots, n$, $\sum_{i=1}^{n} x_i \leq 1$, and $\theta > 0$.

4.12e Inverted Dirichlet: $g_5(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\Gamma\left(\theta + \sum_{i=1}^{n} u_i\right)}{\Gamma(\theta) \prod_{i=1}^{n} \Gamma(u_i)} \times \frac{\prod_{i=1}^{n} x_i^{u_i - 1}}{\left(1 + \sum_{i=1}^{n} x_i\right)^{\theta} + \sum_{i=1}^{n} x_i}$$

where $u_i > 0$, $x_i \geq 0$, $i=1, \ldots, n$, and $\theta > 0$.

4.12f Negative multivariate hypergeometric:

$$g_6(\underset{\sim}{u},\underset{\sim}{x}) = \frac{N!\,\Gamma\left(\sum_{i=1}^{n} u_i\right)}{\prod_{i=1}^{n} x_i!\,\Gamma(N + \sum_{i=1}^{n} u_i)} \times \prod_{i=1}^{n} \frac{\Gamma(x_i + u_i)}{\Gamma(u_i)},$$

where $u_i > 0$, $x_i = 0,1,\ldots,N$, $\sum_{i=1}^{n} x_i = N$, and $N = 1,2,\ldots$.

4.12g Dirichlet compound negative multinomial: $g_7(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\Gamma\left(N + \sum_{i=1}^{n} x_i\right)\Gamma\left(\theta + \sum_{i=1}^{n} u_i\right)\Gamma(N+\theta)}{\prod_{i=1}^{n} x_i!\,\Gamma(N)\Gamma(\theta)\Gamma\left(N + \theta + \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} x_i\right)} \times \prod_{i=1}^{n} \frac{\Gamma(x_i + u_i)}{\Gamma(u_i)}$$

where $u_i > 0$, $x_i = 0,1,\ldots$, $i = 1,\ldots,n$, $\theta > 0$, and $N = 1,2,\ldots$.

4.12h Multivariate logarithmic series distribution: $g_8(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\left(\sum_{i=1}^{n} x_i - 1\right)!}{\log\left(1 + \sum_{i=1}^{n} u_i\right)} \times \left(1 + \sum_{i=1}^{n} u_i\right)^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{u_i^{x_i}}{x_i!}$$

where $u_i > 0$, $x_i = 0,\ldots$, $i = 1,\ldots,n$.

4.12i Multivariate F distribution: $g_9(\underset{\sim}{u},\underset{\sim}{x}) =$

$$\frac{\Gamma(u)\prod_{i=0}^{n}(2u_i)^{u_i}\prod_{i=1}^{n}x_i^{u_i-1}}{2\prod_{i=0}^{n}\Gamma(\lambda_i)(\lambda_o+\sum_{i=1}^{n}\lambda_i x_i)^{\lambda}},$$

where u_i 0, $i=0,1,\ldots,n$, $u = \sum_{i=1}^{n}\lambda_i, x_i \geq 0$, $i=0,1,\ldots,n$.

4.12j Multivariate Pareto distribution:

$$g_{10}(\underset{\sim}{u},\underset{\sim}{x}) = a(a+1)\ldots(a+n-1)\left(\prod_{i=1}^{n}u_i\right)^{-1} \times \left(\sum_{i=1}^{n}\lambda_i^{-1}x_i - n + 1\right)^{-(a+n)}$$

where $x_i > u_i > 0$, $i=1,\ldots,n$, and $a > 0$.

4.12k Multivariate normal distribution with common variance and common covariance: $g_{11}(\underset{\sim}{u},\underset{\sim}{x}) =$

$|\sqrt{2\pi}\,\Sigma|^{-1}e^{-\frac{1}{2}}(\underset{\sim}{x}-\underset{\sim}{u})\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{u})^{-1}$, where Σ is the positive definite covariance matrix with elements σ^2 along the main diagonal and elements $\rho\sigma^2$ elsewhere, $\rho > -1/(n-1)$.

In Theorem 4.13 below we give another method for constructing AP densities from AI densities on $R^n \times R^n$.

4.13 Theorem. Let $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{x}_1$, and $\underset{\sim}{x}_2$ be elements of R^n and let $\underset{\sim}{u} \equiv (\underset{\sim}{u}_1,\underset{\sim}{u}_2)$ and $\underset{\sim}{x} \equiv (\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Let f_1 and f_2 be AI on $R^n \times R^n$ and let g satisfy $g(\underset{\sim}{u},\underset{\sim}{x}) = g(\underset{\sim}{u}\underset{\sim}{\pi},\underset{\sim}{x}\underset{\sim}{\sigma})$ for all $\underset{\sim}{\pi}$, $\underset{\sim}{\sigma} \in S_n \times S_n$. Then $K(\underset{\sim}{u},\underset{\sim}{x}) \equiv K(\underset{\sim}{u}_1,\underset{\sim}{x}_1;\underset{\sim}{u}_2,\underset{\sim}{x}_2) \equiv g(\underset{\sim}{u},\underset{\sim}{x})[f_1(\underset{\sim}{u}_1,\underset{\sim}{x}_1)f_2(\underset{\sim}{u}_2,\underset{\sim}{x}_2)]$ is AP in $(\underset{\sim}{u}_1,\underset{\sim}{x}_1;\underset{\sim}{u}_2,\underset{\sim}{x}_2)$.

Proof. By Theorem 3.5 (ii), $f_1(\underset{\sim}{u}_1,\underset{\sim}{x}_1)f(\underset{\sim}{u}_2,\underset{\sim}{x}_2)$ is AP. The conclusion follows from Proposition 3.6. ||

Theorem 4.13 applies to all of the densities listed above except 4.12 i, j, and k.

Using Theorem 4.14 below we can construct additional examples of AI densities. Recall that a nonnegative function f(x,y) defined on a subset $A \times B$ of $R^1 \times R^1$ is

<u>totally positive of order 2 (TP_2)</u> if $f(x_1,y_1) \times f(x_2,y_2) \geq f(x_1,y_2)f(x_2,y_1)$ for all $x_1 < x_2$ in A and $y_1 < y_2$ in B. HPS (1977) have established the following relationship between TP_2 functions and AI functions.

<u>4.14 Theorem</u>. (i) Let $f(x,y)$ be TP_2. Then $g(x,y) = \prod_{i=1}^{n} f(x_i,y_i)$ is AI.

(ii) Let $f(x,y)$ be a positive set function function. Then $g(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{i=1}^{n} f(\underset{\sim}{x}_i,\underset{\sim}{y}_i)$ is AI.

Thus we may construct AP densities by taking one (or both) of the functions g_i, $i = 1, 2$, in Theorem 4.11 to be the product of TP_2 densities. This corresponds to the case where the components of one or both of the vectors $(\underset{\sim}{X},\underset{\sim}{Y})$ are mutually independent.

Examples of PSA and AP densities can be generated using densities which are Pólya frequency functions of order 2 (PF_2). A nonnegative function $f(x)$ is PF_2 if $\log f(x)$ is concave on $(-\infty,\infty)$. Equivalently, a nonnegative function is PF_2 if $f(x-\theta)$ is TP_2 for x, θ real. Hence the shifted counterpart of a PF_2 density is TP_2.

In Remark 4.15 below we see how a result of Karlin (1968) can be used to construct examples of pairs of random variables (X,Y) having TP_2 densities.

<u>4.15 Remark</u>. Often in reliability situations we are interested in components whose lifelengths have TP_2 densities $f(\theta,x)$ — for example, Poisson or exponential, where θ is a random variable which depends on the environment. Let X and Y be the lifelengths of two independent components with TP_2 densities $F(\theta,x)$ and $g(\theta,y)$, respectively. Let θ have distribution $\pi(\theta)$. Then the joint distribution of (X,Y) is given by $K(x,y) = \int f(\theta,x)g(\theta,y)d\pi(\theta)$. By the composition theorem for TP_2 functions (see Karlin, 1968, p.16), $k(x,y)$ is TP_2 in x and y.

<u>4.16 Theorem</u>. Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be independent, each having a common TP_2 density in (x,y). Then $(\underset{\sim}{X},\underset{\sim}{Y}) \equiv ((X_1,\ldots,X_n), (Y_1,\ldots,Y_n))$ are SSA.

<u>Proof</u>. By Theorem 4.14, $(\underset{\sim}{X},\underset{\sim}{Y})$ has a joint density which is AI and hence, PSA. By Theorem 4.7, $(\underset{\sim}{X},\underset{\sim}{Y})$ are SSA. ||

Examples of PSA and AP densities can also be constructed from Schur functions.

Recall that a vector $\underset{\sim}{x} \in R^n$ is said to <u>majorize</u> a vector $\underset{\sim}{y} \in R^n$ if $\sum_{i=1}^{k} x_{[i]} \geq \sum_{i=1}^{k} y_{[i]}$ for $k = 1,\ldots,n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ where $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$. A function f is said to be <u>Schur-concave</u> (<u>-convex</u>) if $f(\underset{\sim}{x}) \leq (\geq)\ f(\underset{\sim}{y})$ whenever $\underset{\sim}{x}$ majorizes $\underset{\sim}{y}$. The following theorem yields examples of AI functions constructed from Schur functions.

<u>4.17 Theorem</u>. (HPS, 1977) (i) Let $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv h(\underset{\sim}{x}-\underset{\sim}{y})$. Then f is AI if and only if h is Schur-concave.

(ii) Let $f(\underset{\sim}{x},\underset{\sim}{y}) \equiv h(\underset{\sim}{x}+\underset{\sim}{y})$. Then f is AI if and only if h is Schur-convex.

Thus, for Theorem 4.11, we can use Schur-concave densities to construct examples of PSA and AP densities.

In the next subsection we will study models in which observations of certain SSA random vectors $(\underset{\sim}{X},\underset{\sim}{Y})$ are subject to contamination (measurement errors). We will show that if the measurement errors $\underset{\sim}{W}$ and $\underset{\sim}{Z}$, say, have Schur-concave densities then $(\underset{\sim}{X}+\underset{\sim}{W},\ \underset{\sim}{Y}+\underset{\sim}{Z})$ is SSA under some independence assumptions.

MO (1974) give a number of interesting examples of Schur-concave densities.

Joint densities of the form $f(\underset{\sim}{x}) = g(\underset{\sim}{x}\Lambda\underset{\sim}{x}')$ where g is a decreasing function and Λ is a positive definite matrix with $\lambda_{11} = \ldots = \lambda_{mm}$ and $\lambda_{ij} = \lambda$ for $i \neq j$ are called <u>elliptically contoured densities</u>. MO show that elliptically contoured densities are Schur-concave.

Another useful example of a Schur-concave density due to MO is:

Let $X_1, \ldots, X_n$ be independent with PF_2 density f. Then the joint density of $\underset{\sim}{X}$, $f_{\underset{\sim}{X}} = \prod_{i=1}^{n} f(X_i)$ is Schur-concave.

<u>Subsection 4.3 Operations On Random Vectors Which are Stochastically Arranged Like Parameters $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$</u>. In this subsection we show that certain operations on pairs of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ preserve the SA property. First we show that if $(\underset{\sim}{X},\underset{\sim}{Y})$ are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, then so are $(\underset{\sim}{r},(\underset{\sim}{X}),\underset{\sim}{s}(\underset{\sim}{Y}))$, where $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are

rank-like functions. In particular we establish this result for the rank order of random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Next we show that the SA property is preserved under certain contamination models and give examples. We also point out that many of the results of Section 3 can be used to perform operations on random vectors which are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ which preserve the SA property.

In Theorem 4.18 below we show that if the random vectors $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\underset{\sim}{Y}(\underset{\sim}{\beta}))$ are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ and $\underset{\sim}{r}(\underset{\sim}{x})$ and $\underset{\sim}{s}(\underset{\sim}{y})$ are rank-like transformations, then the transformed random vectors $(\underset{\sim}{r}(\underset{\sim}{X}(\underset{\sim}{\alpha})),\ \underset{\sim}{s}(\underset{\sim}{Y}(\underset{\sim}{\beta})))$ are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$.

<u>4.18 Theorem</u>. Let $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\ \underset{\sim}{Y}(\underset{\sim}{\beta}))$ be SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Let $\underset{\sim}{r}$ and $\underset{\sim}{s}$ be either both rank-like or both reverse rank-like transformations. Then $(\underset{\sim}{R}(\underset{\sim}{\alpha}),\underset{\sim}{S}(\underset{\sim}{\beta})) \equiv (\underset{\sim}{r}(\underset{\sim}{X}(\underset{\sim}{\alpha})),\ \underset{\sim}{s}(\underset{\sim}{Y}(\underset{\sim}{\beta})))$ are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$.

To prove Theorem 4.18 we need a lemma.

<u>4.19 Lemma</u>. Let $(\underset{\sim}{X}(\underset{\sim}{\alpha}),\ \underset{\sim}{Y}(\underset{\sim}{\beta}))$ be SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ and let $(\underset{\sim}{R}(\underset{\sim}{\alpha}),\underset{\sim}{S}(\underset{\sim}{\beta})) \equiv (\underset{\sim}{r}(\underset{\sim}{X}(\underset{\sim}{\alpha})),\ \underset{\sim}{s}(\underset{\sim}{Y}(\underset{\sim}{\beta})))$, where $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are either both rank-like or both reverse rank-like transformations. Then

(4.3) $$P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \underset{\sim}{u},\ \underset{\sim}{S}(\underset{\sim}{\beta}) = \underset{\sim}{v} \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}})$$

is AP in $(\underset{\sim}{\alpha},\underset{\sim}{u};\ \underset{\sim}{\beta},\underset{\sim}{v})$ for each fixed pair $(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}})$.

<u>Proof</u>. Let $\underset{\sim}{r}(\underset{\sim}{x})$ and $\underset{\sim}{s}(\underset{\sim}{y})$ be rank-like transformations of $\underset{\sim}{x}$ and $\underset{\sim}{y}$. For $\underset{\sim}{\pi},\ \underset{\sim}{\sigma} \in S_n$, we have:

$$P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \underset{\sim}{u},\ \underset{\sim}{S}(\underset{\sim}{\beta}) = \underset{\sim}{v} \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}} =$$

$$\sum_{\underset{\sim}{\pi}} \sum_{\underset{\sim}{\sigma}} I_{\{\underset{\sim}{r}(\vec{\underset{\sim}{x}}\underset{\sim}{\pi}) = \underset{\sim}{u}\}} I_{\{\underset{\sim}{s}(\vec{\underset{\sim}{y}}\underset{\sim}{\sigma}) = \underset{\sim}{v}\}} P(\underset{\sim}{X}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}\,\underset{\sim}{\pi},\ \underset{\sim}{Y}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}}\underset{\sim}{\sigma} \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}}).$$

Now the function $I_{\{\underset{\sim}{w}=\underset{\sim}{z}\}}$ is clearly AI in $(\underset{\sim}{w},\underset{\sim}{z})$. Hence by Theorem 3.5 (ii), $I_{\{\underset{\sim}{x}=\underset{\sim}{u}\}}\ I_{\{\underset{\sim}{y}=\underset{\sim}{v}\}}$ is AP in $(\underset{\sim}{u},\underset{\sim}{x};\ \underset{\sim}{v},\underset{\sim}{y})$. By Corollary 3.17, $I_{\{\underset{\sim}{r}(\underset{\sim}{x})=\underset{\sim}{u}\}} I_{\{\underset{\sim}{s}(\underset{\sim}{y})=\underset{\sim}{v}\}}$ is AP in $(\underset{\sim}{u},\underset{\sim}{r}(\underset{\sim}{x});\ \underset{\sim}{v},\underset{\sim}{s}(\underset{\sim}{y}))$. By Remark 3.4, $I_{\{\underset{\sim}{r}(\vec{\underset{\sim}{x}}\ \underset{\sim}{\pi}) = \underset{\sim}{u}\}}\ I_{\{\underset{\sim}{s}(\underset{\sim}{y}\ \underset{\sim}{\sigma}) = \underset{\sim}{v}\}}$ is AP in $(\underset{\sim}{u},\underset{\sim}{\pi};\ \underset{\sim}{v},\underset{\sim}{\sigma})$.

Since $(\underset{\sim}{X},\underset{\sim}{Y})$ is SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, it follows from Theorem 3.11 that (4.3) is AP in $(\underset{\sim}{\alpha},\underset{\sim}{u};\ \underset{\sim}{\beta},\underset{\sim}{v})$.

The proof for the case where $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are both reverse rank-like is similar. ||

<u>Proof of 4.18 Theorem</u>. We have:

$$P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\underset{\sim}{\pi},\ \underset{\sim}{S}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}}\underset{\sim}{\sigma} \mid \vec{\underset{\sim}{R}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}},\ \vec{\underset{\sim}{S}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}}) =$$

$$\iint P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\underset{\sim}{\pi},\ \underset{\sim}{S}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}}\underset{\sim}{\sigma} \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{y}}) \times P(\vec{\underset{\sim}{X}} \in d\vec{\underset{\sim}{x}},\ \vec{\underset{\sim}{Y}} \in d\vec{\underset{\sim}{y}} \mid \vec{\underset{\sim}{R}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}},\ \vec{\underset{\sim}{S}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}}).$$

By Lemma 4.19, the integrand is AP. Hence by Theorem 3.12 (iii),
$P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\,\underset{\sim}{\pi},\ \underset{\sim}{S}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}}\,\underset{\sim}{\sigma} \mid \vec{\underset{\sim}{R}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}},\ \vec{\underset{\sim}{S}}(\underset{\sim}{\beta}) = \vec{\underset{\sim}{v}})$ is AP in $(\underset{\sim}{\alpha},\underset{\sim}{\pi};\ \underset{\sim}{\beta},\underset{\sim}{\sigma})$ since it is a mixture of AP functions. ||

The techniques used to prove Theorem 4.18 and Lemma 4.19 yield analogous results for random vectors which are SSA. In Theorem 4.20 we state the counterpart of Theorem 4.18 for SSA random vectors.

<u>4.20 Theorem</u>. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA and let $\underset{\sim}{r}$ and $\underset{\sim}{s}$ be either both rank-like or both reverse rank-like transformations. Then $(\underset{\sim}{R},\underset{\sim}{S}) \equiv (\underset{\sim}{r}(\underset{\sim}{X}),\ \underset{\sim}{s}(\underset{\sim}{Y}))$ are SSA.

For random variables $X_1,\dots,X_n$, denote by R_i the rank of X_i among $X_1,\dots,X_n$. The random vector $\underset{\sim}{R} \equiv (R_1,\dots,R_n)$ is called the <u>rank order</u> of $(X_1,\dots,X_n)$. Theorems 4.18 and 4.20 have the following important corollary

<u>4.21 Corollary</u>. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$). Let $\underset{\sim}{R}$ be the rank order of $\underset{\sim}{X}$ and let $\underset{\sim}{S}$ be the rank order of $\underset{\sim}{Y}$. Then the random vectors $(\underset{\sim}{R},\underset{\sim}{S})$ are SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$).

Next we prove a result which is useful in the study of contamination (measurement error) models. In this type of model we are interested in a random vector $\underset{\sim}{X}$ but are able to observe only a vector $\underset{\sim}{X}+\underset{\sim}{W}$, where $\underset{\sim}{W}$ is a vector representing measurement error.

<u>4.22 Theorem</u>. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ have a density which is PSA (AP with parameters $(\underset{\sim}{\alpha},(\underset{\sim}{\beta})$). Let $\underset{\sim}{W}$, $\underset{\sim}{Z}$ be mutually independent and independent of $(\underset{\sim}{X},\underset{\sim}{Y})$, each having a Schur-concave density. Then $(\underset{\sim}{X}+\underset{\sim}{W},\ \underset{\sim}{Y}+\underset{\sim}{Z})$ is SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$).

<u>Proof</u>. We prove the theorem in the case where $(\underset{\sim}{X},\underset{\sim}{Y})$ has a PSA density g. Denote the densities of $\underset{\sim}{W}$ and $\underset{\sim}{Z}$ by $h_{\underset{\sim}{W}}$ and $h_{\underset{\sim}{Z}}$, respectively. Then the density of $(\underset{\sim}{X}+\underset{\sim}{W},\ \underset{\sim}{Y}+\underset{\sim}{Z})$ is:

$$f(\underset{\sim}{w},\underset{\sim}{z}) = \int g(\underset{\sim}{x},\underset{\sim}{y})h_{\underset{\sim}{W}}(\underset{\sim}{w}-\underset{\sim}{x})h_{\underset{\sim}{Z}}(\underset{\sim}{z}-\underset{\sim}{y})m(d\underset{\sim}{x},d\underset{\sim}{y}).$$

Since $h_{\underset{\sim}{W}}$ and $h_{\underset{\sim}{Z}}$ are Schur-concave, by Theorem 4.17(i), $h_{\underset{\sim}{W}}(\underset{\sim}{w}-\underset{\sim}{x})$ and $h_{\underset{\sim}{Z}}(\underset{\sim}{z}-\underset{\sim}{y})$ are AI functions. The conclusion follows from Theorems 3.9 and 4.7.

In the case where g is an AP density, the result follows in a similar way from Theorem 3.11 and Theorem 4.8 . ||

In a contamination model of the type mentioned in the comments preceeding Theorem 4.22, it is often assumed that the measurement error $\underset{\sim}{W}$ is exchangeable multivariate normal with mean zero or multinomial with equal cell probabilities, or, possibly a vector of independent random variables with common Poisson distribution. In Corollary 4.24 below $\underset{\sim}{W}$ and $\underset{\sim}{Z}$ may have any of these distributions or any of the ones given in Example 4.19 of D'Abadie (1981).

To prove Corollary 4.24 we need the following counterpart of Theorem 3.12 for distributions which are SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$).

<u>4.23 Theorem</u>. Let (Θ,f,μ) be a measure space and for each $\theta \in \Theta$ let $F_\theta(\underset{\sim}{x},\underset{\sim}{y})$ be SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$). Assume that for each $\underset{\sim}{x},\underset{\sim}{y}$, $F_\theta(\underset{\sim}{x},\underset{\sim}{y})$ is measurable as a function of θ. Then $F(\underset{\sim}{x},\underset{\sim}{y}) = \int F_\theta(\underset{\sim}{x},\underset{\sim}{y})\mu(d\theta)$ is SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$).

<u>4.24 Corollary</u>. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ have a density which is PSA (AP with parameters $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$). Let $\underset{\sim}{W}(\underset{\sim}{Z})$ have any of the densities in Examples 4.19a through 4.19g of D'Abadie (1981) or let $W_1,\ldots,W_n$ $(Z_1,\ldots,Z_n)$ be independent Poisson random variables with common intensity parameter λ. Then $(\underset{\sim}{X}+\underset{\sim}{W},\ \underset{\sim}{Y}+\underset{\sim}{Z})$ is SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$).

<u>Proof</u>. Since the multinomial and the multivariate normal random vectors in the hypothesis of the theorem have Schur-concave densities the conclusion follows in the first case directly from Theorem 4.22. In the second case, where one or both of $W_1,\ldots,W_n$ and $Z_1,\ldots,Z_n$ are independent Poisson random variables, the result follows from the first case, Theorem 4.23, and the fact that independent Poisson random variables with common intensity parameter when conditioned on their sum become multinomial with equal cell probabilities. ||

4.27 Remark. As noted at the end of Subsection 4.5, the product of the same PF_2 densities is Schur-concave. Hence in Theorem 4.22 one (or both) of the contaminating random vectors can be a vector of independent PF_2 variables.

In general, many results in Section 3 can be used to construct random vectors which are SSA or SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Theorem 4.23 for mixtures illustrates this. As another example, Theorem 3.15 has the following counterpart for distributions which are SSA (SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$) representing a slight generalization of Theorem 4.22.

4.26 Theorem. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ have density f with respect to a permutation invariant measure m given by $f(\underset{\sim}{x},\underset{\sim}{y}) = \iint g_1(\underset{\sim}{\alpha},\underset{\sim}{x})g_2(\underset{\sim}{\beta},\underset{\sim}{y})h(\underset{\sim}{\alpha},\underset{\sim}{\beta})m(d\underset{\sim}{\alpha},d\underset{\sim}{\beta})$, where g_1 and g_2 are AI and h is PSA. Then $(\underset{\sim}{X},\underset{\sim}{Y})$ are SSA.

4.27 Remark. Theorem 4.26 has a reliability interpretation. Suppose that g_1 and g_2 are densities of component lifelengths. We may think of the parameters $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ as random vectors which depend on the environment. We formalize the notion that $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are dependent by supposing that the *joint density* of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ is PSA. Then by the theorem we have that $(\underset{\sim}{X},\underset{\sim}{Y})$ are SSA.

The next result gives another sufficient condition for the distribution function of a pair of random vectors to be PSA. Recall that (X,Y) is right corner set increasing (RCSI) if $P[X > x, Y > y | X > x'; Y > y']$ is increasing in x' and y' for each fixed x and y. Barlow has shown that this is equivalent to F(x,y) being TP_2. Consequently, the following theorem is true.

4.28 Theorem. Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be RCSI. Then the joint survival function of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$, $\bar{F}(\underset{\sim}{x},\underset{\sim}{y}) = \prod_{i=1}^{n} \bar{F}_i(x_i,y_i)$, is PSA.

Subsection 4.4 A Generalization of a Theorem of HPS. We conclude this section with some generalizations of the work of HPS in the spirit of the previous subsections of this section.

HPS(1977) consider a random vector $\underset{\sim}{X}(\underset{\sim}{\alpha})$ indexed by a parameter $\underset{\sim}{\alpha}$ which has the property that its density $f(\underset{\sim}{\alpha},\underset{\sim}{x})$ with respect to a permutation invariant measure is an AI function. They give many examples of random vectors of interest in statistics having the above property, and show a number of interesting results in nonparametric statistics for this type of random vector. Actually, most of the results in their paper concerning this type of random vector (cf. Theorems 4.1, 4.4, and their corollaries) are true under the weaker hypothesis that

(4.4) $$P_{\underset{\sim}{\alpha}}(\underset{\sim}{X} = \underset{\sim}{\vec{x}}\pi \mid \underset{\sim}{\vec{X}} = \underset{\sim}{\vec{x}}) \text{ is AI in } \underset{\sim}{\alpha} \text{ and } \underset{\sim}{\pi} \text{ for all } \underset{\sim}{x}.$$

(That the latter condition is weaker than the requirement that the density $f(\underset{\sim}{\alpha},\underset{\sim}{x})$ be AI follows from the fact that $P_{\underset{\sim}{\alpha}}(\underset{\sim}{X} = \underset{\sim}{x}\pi \mid \vec{\underset{\sim}{X}} = \vec{\underset{\sim}{x}}) = \dfrac{f(\underset{\sim}{\alpha},\underset{\sim}{x})}{\sum_{\underset{\sim}{\alpha}} f(\underset{\sim}{\alpha},\vec{\underset{\sim}{x}}\alpha)}$, whenever the density f exists.)

We will say that the random vector $\underset{\sim}{X}(\underset{\sim}{\alpha})$ is <u>arrangement</u> <u>increasing</u> (AI) whenever (4.4) is satisfied. We prove the following theorem which yields as a corollary a generalization of Theorem 4.1 of HPS given below in Corollary 4.32 and a useful result concerning AI random vectors.

<u>4.29 Theorem</u>. Let $\underset{\sim}{X}(\underset{\sim}{\alpha})$ be AI and let $\underset{\sim}{R}(\underset{\sim}{\alpha}) \equiv \underset{\sim}{r}(\underset{\sim}{X}(\underset{\sim}{\alpha}))$, where $\underset{\sim}{r}$ is a rank-like function. Then $\underset{\sim}{R}(\underset{\sim}{\alpha})$ is AI.

<u>Proof</u>. By Remark 3.4 and Theorem 3.13,

$$P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\pi \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}) = \sum_{\underset{\sim}{\pi}} I_{\{r(\vec{\underset{\sim}{x}}\pi) = \underset{\sim}{u}\}} P(\vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}\pi \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}) \text{ is AI in } \underset{\sim}{\alpha} \text{ and } \underset{\sim}{u}.$$

Now

$$f_{\underset{\sim}{\alpha},\underset{\sim}{u}}(\underset{\sim}{\pi}) = P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\pi \mid \vec{\underset{\sim}{R}}(\underset{\sim}{\alpha}) = \underset{\sim}{u}) = \int P(\underset{\sim}{R}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}\pi \mid \vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{x}}) \times P(\vec{\underset{\sim}{X}}(\underset{\sim}{\alpha}) \in d\vec{\underset{\sim}{x}} \mid \vec{\underset{\sim}{R}}(\underset{\sim}{\alpha}) = \vec{\underset{\sim}{u}}).$$

By the above the integrand is AI. Since $f_{\underset{\sim}{\alpha},\underset{\sim}{u}}(\underset{\sim}{\pi})$ is a mixture of AI functions, the conclusion follows from Theorem 3.13. ||

The corollary below is easily seen to be an immediate consequence of Theorem 4.29.

<u>4.30 Corollary</u>. Let $\underset{\sim}{X}(\underset{\sim}{\alpha})$ be AI and let $\underset{\sim}{R}(\underset{\sim}{\alpha})$ be the rank order of $\underset{\sim}{X}(\underset{\sim}{\alpha})$. Then $\underset{\sim}{R}(\underset{\sim}{\alpha})$ is AI.

HPS proved this result in the case where $\underset{\sim}{X}(\underset{\sim}{\alpha})$ has a density $f(\underset{\sim}{\alpha},\underset{\sim}{x})$ which is AI.

5. Applications to Statistics of Random Vectors Stochastically Arranged Like Parameters (α,β). The theory developed in Sections 3 and 4 has applications in a number of areas in statistics. In this section we present sample applications.

Subsection 5.1 Stochastic Rearrangement Inequalities. Using our theory we obtain stochastic rearrangement inequalities involving the rearrangement of components of random vectors. We show that these inequalities contain well-known deterministic rearrangement inequalities as special cases.

The first inequality we obtain is a stochastic version of the rearrangement inequality of HLP.

5.1 Theorem. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA. Then for all permutations $\underset{\sim}{\pi}$,

(5.1) $$\sum_{i=1}^{n} X_i Y_i \overset{st.}{\geq} \sum_{i=1}^{n} X_i Y_{\pi(i)} \overset{st.}{\geq} \sum_{i=1}^{n} X_i Y_{n-i+1} .$$

Proof. The result follows from Theorem 4.2 and the fact that the function $f(x,y) \equiv xy$ is AI.||

Theorem 5.1 applies to a large number of pairs of random vectors $(\underset{\sim}{X},\underset{\sim}{Y})$. As an illustration let $\underset{\sim}{X} \equiv (X_1,X_2,\ldots,X_n)$ and $\underset{\sim}{Y} \equiv (Y_1,Y_2,\ldots,Y_n)$ be independently distributed exponential, Poisson, or normal (variance 1) with parameters $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. Then $(\underset{\sim}{X},\underset{\sim}{Y})$ has an AP density by Theorem 4.14 and hence the stochastic inequalities in (5.1) hold for $(\underset{\sim}{X},\underset{\sim}{Y})$. Of course, the inequalities in (5.1) are true for any of the pairs of random vectors $(\underset{\sim}{X},\underset{\sim}{Y})$ given in Section 4 which have PSA or AP densities.

We give an example of how Theorem 5.1 may be applied in reliability theory. This generalizes a result of Derman, Lieberman, and Ross (1972) in the case where two vectors are involved.

5.2 Example. Suppose that we have two stockpiles of n units each, of two different types of components. From these stockpiles we are to construct n systems, each composed of a component of type 1 and a component of type 2 arranged in series.

A component i of type j has a random reliability P_i^j, $j = 1, 2$, $i = 1, \ldots, n$. We assume that $\mathbf{P}^1 \equiv (P_1^1, \ldots, P_n^1)$ and $\mathbf{P}^2 \equiv (P_1^2, \ldots, P_n^2)$ are independent, having AI densities with parameters $\alpha_1 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \ldots \leq \beta_n$, respectively. Then, as we have seen in Section 4, $(\mathbf{P}^1, \mathbf{P}^2)$ are SA like $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. For the assembly which pairs the i^{th} component of type 1 with the $\pi(i)^{th}$ component of type 2, the average reliability of n systems is $\frac{1}{n} \sum_{i=1}^{n} P_i^1 P_{\pi(i)}^2$. Thus inequality (5.1) states that the optimal assembly, in terms of average reliability of the n systems, is achieved when the i^{th} component of type 1 is paired with the i^{th} component of type 2.||

Inequalities for functions of min (x,y) have been obtained by Jurkat and Ryser (1966). They show that for nonnegative n-tuples $\mathbf{x}$ and $\mathbf{y}$,

$$\prod_{i=1}^{n} \min(x_i, y_i) \geq \prod_{i=1}^{n} \min(x_i, y_{\pi(i)}) \geq \prod_{i=1}^{n} \min(x_i, y_{n-1+1})$$

for all $\boldsymbol{\pi} \in S_n$, and

$$\sum_{i=1}^{n} \min(x_i, y_i) \geq \sum_{i=1}^{n} \min(x_i, y_{\pi(i)}) \geq \sum_{i=1}^{n} \min(x_i, y_{n-i+1}).$$

Minc (1971) obtained the following similar rearrangement inequalities for products and sums of max (x,y):

$$\prod_{i=1}^{n} \max(x_i, y_i) \leq \prod_{i=1}^{n} \max(x_i, y_{\pi(i)}) \leq \prod_{i=1}^{n} \max(x_i, y_{n-i+1}),$$

and

$$\sum_{i=1}^{n} \max(x_i, y_i) \leq \sum_{i=1}^{n} \max(x_i, y_{\pi(i)}) \leq \sum_{i=1}^{n} \max(x_i, y_{n-i+1}).$$

By arguments similar to those of Theorem 5.1 we can obtain stochastic versions of each of these inequalities:

<u>5.3 Theorem</u>. Let $(\mathbf{X}, \mathbf{Y})$ be SSA. Then the following stochastic inequalities hold for all permutations $\boldsymbol{\pi}$:

$$(5.2)\qquad \prod_{i=1}^{n} \min(X_i, Y_i) \overset{st.}{\geq} \prod_{i=1}^{n} \min(X_i, Y_{\pi(i)}) \overset{st.}{\geq} \prod_{i=1}^{n} \min(X_i, Y_{n-i+1});$$

$$(5.3)\qquad \prod_{i=1}^{n} \min(X_i, Y_i) \overset{st.}{\geq} \sum_{i=1}^{n} \min(X_i, Y_{\pi(i)}) \overset{st.}{\geq} \sum_{i=1}^{n} \min(X_i, Y_{n-i+1});$$

$$(5.4)\qquad \prod_{i=1}^{n} \max(X_i, Y_i) \overset{st.}{\leq} \prod_{i=1}^{n} \max(X_i, Y_{\pi(i)}) \overset{st.}{\leq} \prod_{i=1}^{n} \max(X_i, Y_{n-i+1});$$

$$(5.5)\qquad \sum_{i=1}^{n} \max(X_i, Y_i) \overset{st.}{\leq} \sum_{i=1}^{n} \max(X_i, Y_{\pi(i)}) \overset{st.}{\leq} \sum_{i=1}^{n} \max(X_i, Y_{n-i+1}).$$

Minc (1971) has shown that if $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are nonnegative real n-tuples then

$$\prod_{i=1}^{n} (x_i + y_i) \leq \prod_{i=1}^{n} (x_i + y_{\pi(i)}) \leq \prod_{i=1}^{n} (x_i + y_{n-i+1})$$

for all permutations $\underset{\sim}{\pi} \in S_n$.

Since $\phi(x,y) = -\prod_{i=1}^{n}(x_i + y_i)$ is an AI function, from Theorem 4.2 we obtain the following stochastic version of the above inequalities.

<u>5.4 Theorem</u>. Let $(\underset{\sim}{X}, \underset{\sim}{Y})$ be SSA. Then

$$\prod_{i=1}^{n} (X_i + Y_i) \overset{st.}{\leq} \prod_{i=1}^{n} (X_i + Y_{\pi(i)}) \overset{st.}{\leq} \prod_{i=1}^{n} (X_i + Y_{n-i+1})$$

for all permutations $\underset{\sim}{\pi}$.

London (1970) generalized the results of HLP and Jurkat and Ryser to obtain rearrangement inequalities for sums and products of functions having some convex properties.

Let $x > 0$ and $y \geq 0$. Define $\tilde{f}(x,y) \equiv f(1 + {}^{y}/x)$ where $f(e^z)$ is convex for $z \geq 0$ and $f(1) \leq f(z)$ for $z \geq 1$. Define $\tilde{g}(x,y) \equiv g({}^{y}/x)$ where $g(z)$ is convex for $z \geq 0$ and $g(0) \leq g(z)$ for $z \geq 0$. London has shown that $\tilde{f}$ and $\tilde{g}$ are positive set functions. Applying Theorem 4.2 we can obtain stochastic rearrangement inequalities for sums involving functions of the form $\tilde{f}$ and $\tilde{g}$.

5.5 Theorem. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA. Let f and g satisfy the conditions stated above.

Then
$$\sum_{i=1}^{n} f(1+{}^{Y_i}/X_i) \overset{st.}{\leq} \sum_{i=1}^{n} f(1+{}^{Y_{\pi(i)}}/X_i) \overset{st.}{\leq} \sum_{i=1}^{n} f(1+{}^{Y_{n-i+1}}/X_i)\ ,$$

and
$$\sum_{i=1}^{n} g({}^{Y_i}/X_i) \overset{st.}{\leq} \sum_{i=1}^{n} g({}^{Y_{\pi(i)}}/X_i) \overset{st.}{\leq} \sum_{i=1}^{n} g({}^{Y_{n-i+1}}/X_i)\,.$$

As an example of a function statisfying the conditions on f, take $f(z) \equiv \log(z+1)$. The function $g(z) \equiv z$ satisfies the conditions on g. Another example is the function $g(z) = z\log(z+1)$.

There are many other examples of deterministic rearrangement inequalities involving AI functions in the literature. Using our theory we can obtain stochastic versions of all of these inequalities in a unified way.

5.6 Remark. We note that the stochastic rearrangement inequalities we obtain contain as special cases their deterministic counterparts. Let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be vectors of nonnegative numbers. Let $A = \{\vec{\underset{\sim}{x}}\underset{\sim}{\pi} : \underset{\sim}{\pi} \in S_n\}$ and $B = \{\vec{\underset{\sim}{y}}\underset{\sim}{\pi} : \underset{\sim}{\pi} \in S_n\}$. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be degenerate at $(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}})$. Then $(\underset{\sim}{X},\underset{\sim}{Y})$ has a PSA density with respect to the counting measure on $A \times B$, and, clearly, Corollary 4.3 yields for any AI function f, $f(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}}) \geq f(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}}\underset{\sim}{\pi}) \geq f(\vec{\underset{\sim}{x}},\vec{\underset{\sim}{y}})$, the deterministic rearrangment inequality.

Subsection 5.2 Applications to Tests of Hypothesis. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA like parameters $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Let α_o be a fixed vector of R^n in the orbit of $\underset{\sim}{\alpha}$. In this subsection we have developed to study the problem of testing the hypothesis

(5.6) $$H_o : \underset{\sim}{\beta} = \alpha_o \text{ against } H_a : \underset{\sim}{\beta} \overset{s}{\neq} \underset{\sim}{\alpha_o}\ .$$

Let f be an AI function and define the test T_f by

(5.7) $T_f(\underset{\sim}{x},\underset{\sim}{y}) = 1$ if $f(\underset{\sim}{x},\underset{\sim}{y}) < \nu_{\alpha j} = \gamma$ if $f(\underset{\sim}{x},\underset{\sim}{y}) = \nu_{\alpha j} = 0$ otherwise.

The null hypothesis is rejected with probability $T_f(\underset{\sim}{x},\underset{\sim}{y})$ if $(\underset{\sim}{x},\underset{\sim}{y})$ is observed. Note that in general the numbers ν_α and $\gamma(0<\gamma<1)$ are determined to give size α to the test.

Let $B_{T_f}(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ be the power function of the above test against alternatives $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, that is, $B_{T_f}(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = ET_f(X(\alpha),Y(\underset{\sim}{\beta}))$. We shall need the following definition (see Barlow et al., 1972, Chapter 6).

<u>5.7 Definition</u>. Let $(\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o) \in R^n \times R^n$ be given. A test T has <u>isotonic power</u> against alternative $(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \overset{a}{\leq} (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o)$ (with respect to the ordering "$\overset{a}{\leq}$") if for any $(\underset{\sim}{\alpha}_1,\underset{\sim}{\beta}_1)$ and $(\underset{\sim}{\alpha}_2,\underset{\sim}{\beta}_2)$ in $R^n \times R^n$ such that $(\underset{\sim}{\alpha}_2,\underset{\sim}{\beta}_2) < (\underset{\sim}{\alpha}_1,\underset{\sim}{\beta}_1) < (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o)$ we have $B_{T_f}(\underset{\sim}{\alpha}_2,\underset{\sim}{\beta}_2) \geq B_{T_f}(\underset{\sim}{\alpha}_1,\underset{\sim}{\beta}_1)$.

<u>5.8 Remark</u>. It is a consequence of Definition 5.7 that any test T which is isotonic with respect to the "$\overset{a}{\leq}$" ordering is unbiased for testing

(5.8) $\quad H_o\colon (\alpha_o,\beta) \overset{a}{=} (\alpha_o,\beta_o)$ against $H_a\colon (\alpha_o,\beta) < (\alpha_o,\beta_o)$, $(\alpha_o,\beta) \overset{a}{\neq} (\alpha_o,\beta_o)$.

Note that by Remark 2.2, the hypotheses in (5.8) are equivalent to those in (5.6).

It follows from Theorem 3.10, that tests of the form given in (5.7) are isotonic with respect to the arrangement ordering and, consequently, by Remark 5.8 such that tests will be unbiased for testing H_o against H_a. We state this formally in the theorem that follows.

<u>5.9 Theorem</u>. Let $(\underset{\sim}{X},\underset{\sim}{Y})$ be SSA like $(\underset{\sim}{\alpha}_o,\underset{\sim}{\beta})$. Consider testing the hypothesis

$$H_o\colon (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}) \overset{a}{=} (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o) \quad \text{against} \quad H_a\colon (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}) \overset{a}{<} (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o),\ (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}) \overset{a}{\neq} (\underset{\sim}{\alpha}_o,\underset{\sim}{\beta}_o).$$

Let f be an AI function and let T_f be the test given in (5.7). Then the test T_f has isotonic power against alternatives $(\underset{\sim}{\alpha}_o,\beta) \overset{a}{\leq} (\alpha_o,\beta_o)$. Consequently, a test based on T_f is unbiased for testing $H_o\colon \underset{\sim}{\alpha}_o \overset{s}{=} \underset{\sim}{\beta}$ against $H_a\colon \underset{\sim}{\alpha}_o \overset{s}{\neq} \underset{\sim}{\beta}$.

A number of well-known statistics are AI functions and hence can be used to test the hypotheses in (5.6).

5.10 Example. The following statistics are AI functions:

1. Product moment correlation: Pearson's product moment correlation coefficient is given by

$$r = \frac{\sum_{i,j} (x_i - x_j)\ (y_i - y_j)}{\left\{ \sum_{i,j} (x_i - x_j)^2 \sum_{i,j} (y_i - y_j)^2 \right\}^{1/2}}$$

2. Spearman's ρ: Spearman's ρ is given by

$$\rho = \frac{\sum_{i,j} (r_i - r_j)\ (s_i - s_j)}{\sum_{i,j} (r_i - r_j)^2}$$

where r_i is the rank of x_i and s_i is the rank of y_i. Spearman's ρ can be viewed as the sample correlation coefficient computed for the ranks.

3. Kendall's τ: Kendall's correlation coefficient τ is given by

$$\tau = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \xi(x_i,x_j)\xi(y_i,y_j) \quad \text{where} \quad \xi(a,b) = \begin{cases} 1, & \text{if } (a-b) \geq 0 \\ -1, & \text{if } (a-b) < 0 \end{cases}$$

4. A general correlation coefficient of Daniels: Daniels (1948) offers the following quantity as a general measure of correlation where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ may be either the observations or their ranks:

$$\delta = \frac{\sum_{i,j} a(x_i,x_j)b(y_i,y_j)}{\left\{ \sum_{i,j} a^2(x_i,x_j) \sum_{i,j} b^2(y_i,y_j) \right\}^{1/2}}$$

where $a(x_i,x_j) = -a(x_j,x_i)$, $b(y_i,y_j) = -b(y_j,y_i)$ and $a(x_i,x_j)$, $b(y_i,y_j)$ are non-decreasing with increasing rank separation.

5. The quadrant test: Lex $\underset{\sim}{x},\underset{\sim}{y}$ be either the observations or their ranks. Let $\underset{\sim}{x}_{med}$ ($\underset{\sim}{y}_{med}$) denote the median of $x_1, \ldots, x_n$ ($y_1, \ldots, y_n$). The quadrant statistic is the sum

$$\mu = \sum_{i=1}^{n} a(x_i)b(y_i), \quad \text{where}$$

(5.9)
$$a(x_i) = \begin{cases} 0, \text{ if } x_i \leq \underset{\sim}{x}_{med} \\ 1, \text{ if } x_i > \underset{\sim}{x}_{med} \end{cases}$$

and $b(y_i)$ is similarly defined.

6. Blomquist's quadrant test for positive association: Blomquist proposed the following test for positive association:

$$\beta = \sum_{i=1}^{n} [a_1(x_i)b_1(y_i) + a_2(x_i)b_2(y_i)]$$

where a_1 and b_1 are defined in (5.9), $a_2 = 1-a_1$, and $b_2 = 1-b_1$.

7. Scores tests of Bhuchongkul: Bhuchongkul (1964) proposed a class of rank tests based on statistics of the form

$$\sigma = \sum_{i=1}^{n} A(r_i)B(s_i)$$

where A and B are nondecreasing functions.

The proofs are omitted.

From the fact that the product moment correlation is AI, it follows that a test of H_o against H_a based on r is equivalent to a test of H_o against H_a based on the statistic $\sum_{i=1}^{n} x_i y_i$ which is also AI. It also follows that a test based on ρ is equivalent to a test based on $\sum_{i=1}^{n} r_i s_i$.

Daniels (1948) correlation coefficient includes as special cases, r, ρ and τ by appropriate choices of a and b.

The statistic σ in (7) above reduces to Spearman's ρ by taking $A(i) = B(i) = i$. This test also includes the normal scores test of Fisher-Yates by $A(r_i)(B(s_i))$ to be the r_i-th (s_i-th) standard normal order statistic.

5.11 Example. Some further examples of statistics which are AI functions are given below. They are due to Savage (1957).

1. $T_1(1, \ldots, n; r_1, \ldots, r_n) = \sum_{i=1}^{n} i\, r_i$

2. $T_2(k, \ldots, n; r_1, \ldots, r_n) = \prod_{i=1}^{n} (r_1 + \ldots + r_i)^{-1}$

3. $T_3(1, \ldots, n; r_1, \ldots, r_n) = \sum_{i,j=1}^{n} d(r_i, r_j)$,

   where $d(a,b) = 1$ if $a < b$, 0 if $a \geq b$

4. $T_4(1, \ldots, n; r_1, \ldots, r_n) = \sum_{n+1}^{2n} d(n, r_i)$

5. $T_5(E_1, \ldots, E_n; r_1, \ldots, r_n) = \sum_{i=1}^{n} E_i\, r_i$

6. $T_6(B_1, \ldots, B_n; r_1, \ldots, r_n) = \sum_{i=1}^{m} B_{r_i}$, $m = 1, \ldots, n$.

Next we give some sample applications of Theorem 5.9 to contingency table analysis.

5.12 Application. Suppose that we ask a group of people selected at random to indicate their preference for one among k different objects. If the group consists of males (Group 1) and females (Group 2), say, we may be interested in whether or not these two groups have similarly arranged preferences. Conditioning on n, the total number of persons in the group, the numbers of preferences (cell frequencies) $n_{11}, n_{12}, \ldots, n_{1k}, n_{21}, n_{22}, \ldots, n_{2k}$ $\left(\sum_{i=1}^{2} \sum_{j=1}^{k} n_{ij} = n \right)$ may be interpreted as an observation from a multinomial distribution $K(\underset{\sim}{p}_1, \underset{\sim}{n}_1; \underset{\sim}{p}_2, \underset{\sim}{n}_2)$ with probabilities $p_{11}, p_{12}, \ldots, p_{1k}, p_{21}, p_{22}, \ldots, p_{2k}$. $\left(\sum_{i=1}^{2} \sum_{j=1}^{k} p_{ij} = 1 \right)$. Suppose that the preferences of one of the groups, Group 1, say, are known, that is, the vector $(p_{11}, \ldots, p_{1k})$ is a known, fixed vector. We are interested in testing the hypothesis

$$H_o: (p_{11}, p_{12}, \ldots, p_{1k}) \overset{s}{=} (p_{21}, p_{22}, \ldots, p_{2k})$$

against

$$H_a: (p_{11}, p_{12}, \ldots, p_{1k}) \overset{s}{\neq} (p_{21}, p_{22}, \ldots, p_{2k}).$$

As we have seen in Theorem 5.2, $K(\underset{\sim}{p}_1,\underset{\sim}{n}_1;\underset{\sim}{p}_2,\underset{\sim}{n}_2)$ is AP, where $\underset{\sim}{p}_1 = (p_{11},p_{12},\ldots,p_{1k})$, $p_2 = (p_{21},p_{22},\ldots,p_{2k})$, $\underset{\sim}{n}_1 = (n_{11},n_{12},\ldots,n_{1k})$, and $\underset{\sim}{n}_2 = (n_{21},n_{22},\ldots n_{2k})$. Thus by 5.9 a test of H_o against H_a is unbiased if the test statistic is an AI function. For example, the test which rejects $\sum_{i=1}^{k} n_{1i}n_{2i} < c$ for an appropriate c is unbiased for testing H_o against H_a.

5.13 Application. A similar result follows if we condition jointly on n_1, the total number of males, and on n_2, the total number of females (the fixed row totals case in contingency tables.) It is straightforward to show that $K(\underset{\sim}{p}_1,\underset{\sim}{n}_1; \underset{\sim}{p}_2,\underset{\sim}{n}_2) = M(\underset{\sim}{p}_1,\underset{\sim}{n}_1)M(\underset{\sim}{p}_2,\underset{\sim}{n}_2)$, the product of two multinomial densities. As we have seen in Theorem 4.12, K is AP. Hence, in this case, we also have that a test of H_o against H_a based on an AI test statistic is unbiased.

5.14 Remark. In Theorem 4.18 we showed that if $(\underset{\sim}{X},\underset{\sim}{Y})$ are SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ then their rank order $(\underset{\sim}{R},\underset{\sim}{S})$ is SA like $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Thus Theorem 5.9 also holds for test statistics T_f based on the rank order of $(\underset{\sim}{X},\underset{\sim}{Y})$. A useful application of the above remark arises in testing for the existence of positive dependence between two time series. An example is described below.

5.15 Application. Studies of air pollution have shown that automobile exhaust is the major source of lead elemental air pollution in many urban areas. It is believed that automobile exhaust is also the major source of bromine pollution in the atmosphere. For a particular city, we wish to determine whether automobile exhaust is the predominant source of both of these two pollutants or, alternatively, whether other sources are responsible for bromine pollution. Suppose that λ_i, the concentration of lead at time i, $i = 1, \ldots, n$, is known. Let $\underset{\sim}{\lambda}_o = (\lambda_1,\ldots,\lambda_n)$.

To help in distinguishing between the two alternative hypotheses, we test H_o: $\underset{\sim}{\lambda}_o \overset{s}{=} \underset{\sim}{\beta}$ against H_a: $\underset{\sim}{\lambda}_o \overset{s}{\neq} \underset{\sim}{\beta}$, where β_i is the true concentration of bromine at time i, i = 1, ..., n, and $\beta = (\beta_1, \ldots, \beta_n)$. Rejection of H_o would indicate that sources other than automobile exhaust contribute to the bromine pollution.

Observations $\underset{\sim}{L}$ on lead and $\underset{\sim}{B}$ on bromine are assumed to be governed by a joint AP density with parameters $(\underset{\sim}{\lambda}_o, \underset{\sim}{\beta})$. By Theorem 5.9, we conclude that a test using <u>an AI test statistic based on the ranks of L and B is isotonic and is, consequently, unbiased against H_a</u>. Nonparametric tests for this type of co-movement between time series have been proposed by Moore and Wallis (1943) and Goodman and Grunfeld (1961).

<u>5.16 Remark</u>. Suppose that the measurement $\underset{\sim}{L}$ and $\underset{\sim}{B}$ are subject to errors $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ with $\underset{\sim}{X} \sim \text{MVN}(\underset{\sim}{0}, \Sigma(\rho_1))$ and $\underset{\sim}{Y} \sim \text{MVN}(\underset{\sim}{0}, \Sigma(\rho_2))$, where

$$\Sigma(\rho) = \begin{pmatrix} \sigma^2 & & \rho\sigma^2 \\ & \ddots & \\ \rho\sigma^2 & & \sigma^2 \end{pmatrix}$$

for $0 \le \rho \le 1$. Since the density of both $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ is Schur-concave by Corollary 4.24, $(\underset{\sim}{L} + \underset{\sim}{X}, \underset{\sim}{B} + \underset{\sim}{Y})$ is SA like $(\underset{\sim}{\lambda}, \underset{\sim}{\beta})$ and, as before, a test using an AI test statistic based on the ranks of $\underset{\sim}{L} + \underset{\sim}{X}$ and $\underset{\sim}{B} + \underset{\sim}{Y}$ is isotonic.

A similar result may be obtained for the examples given above involving contingency tables. In this case we suppose that the observations $\underset{\sim}{N}_1$ and $\underset{\sim}{N}_2$, say, are subject to errors $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ each having a Poisson distribution. By Corollary 4.24 $\underset{\sim}{N}_1 + \underset{\sim}{X}$ and $\underset{\sim}{N}_2 + \underset{\sim}{Y}$ have AP densities.

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